SOLUTION HW #5

3.4.5 By (3.3.13),

$$f(x) = \cos\frac{\pi x}{L} \sim \sum_{n=\text{even}} B_n \sin\frac{n\pi x}{L}, \quad 0 \le x \le L, \quad \text{where } B_n = \frac{4n}{\pi(n^2-1)}, \ (n \text{ even})$$

$B_n = 0$ if n odd

By (3.4.13),

$$f'(x) = -\frac{\pi}{L}\sin\frac{\pi x}{L} \sim \frac{1}{L}(-1-1) + \sum_n \left(\frac{n\pi}{L}B_n + \frac{2}{L}\left((-1)^{n+1}-1\right)\right)\cos\frac{n\pi x}{L}$$

$$= -\frac{2}{L} + \sum_{n=\text{even}} \left(n\frac{\pi}{L}B_n - \frac{2}{L}\right)\cos\frac{n\pi x}{L}$$

$\Rightarrow$

$$\sin\frac{\pi x}{L} \sim \frac{2}{\pi} + \sum_{n=\text{even}}\left(\frac{2}{\pi} - nB_n\right)\cos\frac{n\pi x}{L} = \frac{2}{\pi}\sum_{n=\text{even}} A_n \cos\frac{n\pi x}{L}, \quad A_n = -\frac{2(n^2+1)}{n^2-1}$$

3.4.6 The second differentiation is NOT VALID since $e^0 \neq 0$, $e^L \neq 0$

Using 3.4.13 we obtain:

$$f(x) = e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin\frac{n\pi x}{L}$$

$$f'(x) = e^x = \frac{1}{L}(e^L - 1) + \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty}\frac{2}{L}\left((-1)^n e^L - 1\right)\cos\frac{n\pi x}{L}$$

Here $A_0 = \frac{1}{L}\int_0^L e^x\,dx = \frac{e^L - 1}{L}$ ok

$$A_n = \left(\frac{n\pi}{L}\right)^2 A_n + \frac{2}{L}\left((-1)^n e^L - 1\right) \Rightarrow A_n = \left(1 - \left(\frac{n\pi}{L}\right)^2\right)^{-1}\frac{2}{L}\left((-1)^n e^L - 1\right)$$

4.2.1

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0(x)\frac{\partial^2 u}{\partial x^2} + Q(x,t)\rho_0(x) \quad (1)$$

if $\frac{\partial u}{\partial t} = 0$, and $Q(x,t) = -g$, then $T_0\frac{\partial^2 u}{\partial x^2} = g\rho_0(x) \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\rho_0}{T_0}g$

(a) Assume ρ_0, T_0 constant $\Rightarrow u(x) = \frac{1}{2}\frac{\rho_0}{T_0}gx^2 + c_1 x + c_2$

$u(0) = 0 \Rightarrow c_2 = 0$

$u(L) = \frac{1}{2}\frac{\rho_0}{T_0}gL^2 + c_1 L = 0$

$\Rightarrow \frac{1}{2}\frac{\rho_0}{T_0}gL + c_1 = 0 \Rightarrow c_1 = -\frac{1}{2}\frac{\rho_0 g L}{T_0}$

Hence $\boxed{u_E(x) = \frac{1}{2}\frac{\rho_0}{T_0}gx^2 - \frac{1}{2}\frac{\rho_0 g}{T_0}Lx}$

(b) Let $v(x,t) = u(x,t) - u_E(x)$.

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 u}{\partial t^2}; \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_E}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\rho_0}{T_0}g$$

By 4.2.7, $\frac{\partial^2 v}{\partial t^2} = \frac{\partial u^2}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2 u}{\partial x^2} - g \Rightarrow \boxed{\frac{\partial v^2}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2 v}{\partial x^2}} = \frac{T_0}{\rho_0}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\rho_0}{T_0}g\right) = c^2\frac{\partial^2 v}{\partial x^2}$

4.2.2 $c^2 = \frac{T_0}{\rho_0} = \frac{\left[kg\,\frac{m}{s^2}\right]}{[kg/m]} = \left[\frac{m^2}{s^2}\right]$ it is dimensionally a VELOCITY SQUARED

4.4.1

e-functions are $\sin\frac{n\pi ct}{L}$, $\cos\frac{n\pi ct}{L}$

(a) VIBRATING STRING, Fixed ends

CIRCULAR FREQ. (# oscillations in 2π unit time) is

$$\frac{n\pi c}{L}, \quad n = 1, 2, 3 \ldots$$

FREQUENCIES (# oscillations per unit time)

$$\frac{nc}{2L}$$

(b) VIBRATING STRING $u(0,t) = 0$, $\frac{\partial u}{\partial x}(H,t) = 0$

SPACE-DEP. EQ. IS $\varphi''(x) = -\lambda\varphi(x)$ + $\varphi(0) = 0$, $\varphi'(H) = 0$

$$\Rightarrow \varphi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}\,x$$

$$\varphi(0) = c_1 = 0$$

$$\varphi'(x) = -c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x$$

$$\varphi'(H) = 0 \iff \cos\sqrt{\lambda}H = 0 \Rightarrow \sqrt{\lambda}H = (2n-1)\frac{\pi}{2}$$

$$\Rightarrow \lambda = \left(\frac{(2n-1)\frac{\pi}{2}}{H}\right)^2$$

e-Functions are of the Form:

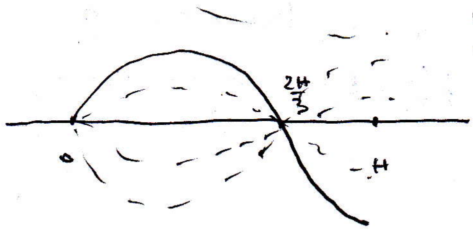
$$\sin\frac{(2n-1)\frac{\pi}{2}}{H}x \;\sin\frac{(2n-1)\frac{\pi}{2}}{H}ct, \quad \sin\frac{(2n-1)\frac{\pi}{2}}{H}x\;\cos\frac{(2n-1)\frac{\pi}{2}}{H}ct$$

$$= \frac{1}{2}\cos\frac{(2n-1)\pi}{2H}(x-ct) - \frac{1}{2}\cos\frac{2n-1}{2H}\pi(x+ct)$$

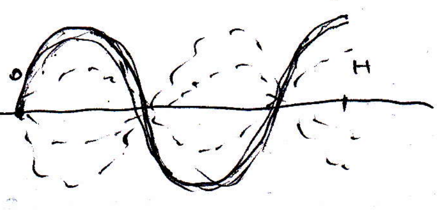
CIRCULAR FREQ.

$$\frac{(2n-1)\pi c}{2H}$$

$\underline{n=1}$ $\sin\frac{\pi}{2H}x$



$\underline{n=2}$ $\sin\frac{3\pi}{2H}x$



$\underline{n=3}$ $\sin\frac{5\pi x}{2H}$

(c) IF $H = L/2$ in part (a) and $n \to 2n-1$, the CIRC. FREQ, the CIRC. FREQ. For part (a) becomes

$$\frac{(2n-1)\pi c}{2H}$$