HW 6 SOLUTION

7.3.4 $u_{tt} = c^2(u_{xx}+u_{yy}) \qquad 0 \le x \le L, \quad 0 \le y \le H$

$u(x,y,0)=0, \quad u_t(x,y,0)=\alpha(x,y)$

5 Pts (a) $u(0,y,t)=u(L,y,t)=0, \quad u_y(x,0,t)=u_y(x,H,t)=0$

Using separable solution $u(x,y,t)=f(x)g(y)h(t)$ as in textbook/class, we obtain the 3 equations:

(1) $\dfrac{d^2h}{dt^2} = -\lambda c^2 h$

(2) $\dfrac{d^2f}{dx^2} = -\mu f \qquad$ B.C. $f(0)=f(L)=0$

(3) $\dfrac{d^2g}{dy^2} = -(\lambda-\mu) g \qquad g'(0)=g'(H)=0$

Solving eq. (2) as in class/book, we obtain

e-values $\mu_n = \left(\dfrac{n\pi}{L}\right)^2, \quad n=1,2,3,\dots$

e-functions $f_n(x) = \sin\dfrac{n\pi x}{L}$

To solve eq. (3), for each n, we look at eq.

$\dfrac{d^2g}{dy^2} = -(\lambda_n-\mu_n)g$

Solutions are of the form $g(y) = A\cos\sqrt{\lambda_n-\mu_n}\,y + B\sin\sqrt{\lambda_n-\mu_n}\,y$

NOTE $g'(y) = -A\sqrt{\lambda_n-\mu_n}\sin\sqrt{\lambda_n-\mu_n}\,y + B\sqrt{\lambda_n-\mu_n}\cos\sqrt{\lambda_n-\mu_n}\,y$

$g'(0) = B\sqrt{\lambda_n-\mu_n} = 0 \Rightarrow B=0$

$g'(H) = -A\sqrt{\lambda_n-\mu_n}\sin\sqrt{\lambda_n-\mu_n}\,H = 0$

Hence it must be $\sqrt{\lambda_n-\mu_n}\,H = m\pi$

$\Leftrightarrow \lambda_{nm}-\mu_n = \left(\dfrac{m\pi}{H}\right)^2, \quad m=0,1,2,3.$

$\Rightarrow \lambda_{nm} = \left(\dfrac{n\pi}{L}\right)^2 + \left(\dfrac{m\pi}{H}\right)^2$

e-functions are $g_m(y) = \cos\dfrac{m\pi y}{H}$ ← NOTE: for $m=0$, g_0 is a NONTRIVIAL e-function

Solving (1) with λ_{nm} found above we have

$h(t)$: lin. combination of $\sin c\sqrt{\lambda_{nm}}\,t$, $\cos c\sqrt{\lambda_{nm}}\,t$

Hence:

$$\boxed{u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \cancel{A_{nm}\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}\cos c\sqrt{\lambda_{nm}}\,t} + B_{nm}\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}\sin c\sqrt{\lambda_{nm}}\,t}$$

↑ VANISHES by I.C.

to satisfy I.C.

$$u(x,y,0) = \sum_m \sum_n A_{nm} \sin\frac{n\pi x}{L} \cos\frac{m\pi y}{H} = 0$$

$$\Rightarrow A_{nm} = 0$$

$$u_t(x,y,0) = c\sqrt{\lambda_{nm}} \sum_m \sum_n B_{nm} \sin\frac{n\pi x}{L} \cos\frac{m\pi y}{H} = \alpha(x,y)$$

Here $c\sqrt{\lambda_{nm}}\, B_{nm} = \frac{2}{L}\int_0^L \frac{2}{H}\int_0^H \alpha(x,y) \cos\frac{m\pi y}{H}\, dy \, \sin\frac{n\pi x}{L}\, dx$

(b) We proceed as in (a) with the difference that eq. (2) is replaced by

5pts

(2') $\frac{d^2 f}{dx^2} = -\mu f$ B.C. $f'(0) = f'(L) = 0$

Solutions are of the form

$$f(x) = A\cos\sqrt{\mu}\,x + B\sin\sqrt{\mu}\,y$$

$$f'(0) = B\sqrt{\mu} = 0 \Rightarrow B = 0$$

$$f'(L) = -A\sqrt{\mu}\sin\sqrt{\mu}L = 0 \Rightarrow \boxed{\mu_n = \left(\frac{n\pi}{L}\right)^2} \quad n = 0,1,2 - -$$

e-values

e-functions $\boxed{f_n(x) = \cos\frac{n\pi}{L}}$

Eq. (3) is solved as in (a) giving:

e-values $\lambda_{nm} - \mu_n = \boxed{\left(\frac{m\pi}{H}\right)^2}$ $m = 0,1,2,\dots$

e-functions $\boxed{g_m(y) = \cos\frac{m\pi}{H}}$

To solve eq. (1) we NEED to consider the case $\lambda_{nm} \neq 0$ and $\lambda_{nm} = 0$

If $\lambda_{nm} \neq 0$ $(n \neq 0$ or $m \neq 0)$ then $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$ and

$$h(t) = K_1 \cos c\sqrt{\lambda_{nm}}\,t + K_2 \sin c\sqrt{\lambda_{nm}}\,t$$

If $\lambda_{00} = 0$, then direct solution of eq. (1) yields

$$h(t) = c_1 t + c_2$$

Since $u(x,y,0) = 0$ implies $h(0) = 0$, then it must be $c_2 = 0$

Using principle of superposition we conclude that

$$\boxed{u(x,y,t) = \sum_{(m,n)\neq(0,0)}^{\infty} \sum^{\infty} \underbrace{\cancel{A_{nm} \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H}}}_{\text{vanishes by I.C.}} \overset{\text{constant}}{+} B_{nm} \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H} \sin c\sqrt{\lambda_{nm}}\,t + A_{00}\, t}$$

As in part (a), I.C. implies $A_{nm} = 0$

and $\cancel{A_{nm} = \frac{1}{LH}\int_0^L \int_0^H \alpha(x,y) \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H}\, dx\, dy}$

$$u_t(x,y,0) = \sum c\sqrt{\lambda_{nm}}\, B_{nm} \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H} + A_{00} = \alpha(x,y)$$

Here $B_{nm} = \frac{1}{c\sqrt{\lambda_{nm}}} \frac{1}{LH}\int_0^L \int_0^H \alpha(x,y) \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{L}\, dx\, dy$ if $(n,m) \neq (0,0)$

$$A_{00} = \frac{1}{LH}\int_0^L \int_0^H \alpha(x,y)\, dx\, dy$$