①

By (2.7.46), when $\frac{\partial u}{\partial t}(r,\theta,0)=0$, the general solution of wave equation in circular membrane is of form

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\cos m\theta \cos c\sqrt{\lambda_{mn}}\, t$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin m\theta \cos c\sqrt{\lambda_{mn}}\, t$$

$$u(r,\theta,0)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\cos m\theta +$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin m\theta$$

Since $u(r,\theta,0)=\alpha(r)\cos 2\theta$, then

$$B_{mn}=0 \ \forall m,n$$
$$A_{2n}\neq 0, \quad A_{mn}=0 \text{ if } m\neq 2$$

Here

$$\boxed{u(r,\theta,t)=\sum_{n=1}^{\infty} A_{2n} J_2(\sqrt{\lambda_{2n}}\, r)\cos 2\theta \cos c\sqrt{\lambda_{2n}}\, t}$$

with
$$A_{2n}=\frac{\iint \alpha(r) J_2(\sqrt{\lambda_{2n}}\, r)\cos^2 2\theta \, r\,dr\,d\theta}{\iint (J_2(\sqrt{\lambda_{2n}}\, r)\cos 2\theta)^2\, r\,dr\,d\theta}$$
$$=\boxed{\frac{\int_0^a J_2(\sqrt{\lambda_{2n}}\, r)\,\alpha(r)\, r\,dr}{\int_0^a (J_2(\sqrt{\lambda_{2n}}\, r))^2\, r\,dr}}$$

$\boxed{\text{HW 7}}$

(2) ~~Ex 7.7.1~~

By (7.7.45), solution of wave equation in this setting is of the form

$$J_m(\sqrt{\lambda_{mn}}\, r)\begin{Bmatrix}\cos m\theta \\ \sin m\theta\end{Bmatrix}\begin{Bmatrix}\cos c\sqrt{\lambda_{mn}}\, t \\ \sin c\sqrt{\lambda_{mn}}\, t\end{Bmatrix}$$

if $u(r,\theta,0)=0$, then term $\cos c\sqrt{\lambda_{mn}}\, t$ in expression above will not be needed.

Hence general solution is of the form:

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\cos m\theta \sin c\sqrt{\lambda_{mn}}\, t + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin m\theta \sin c\sqrt{\lambda_{mn}}\, t.$$

$$\frac{\partial u}{\partial t}(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}}\, A_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\cos m\theta + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}}\, B_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin m\theta$$

Since $\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 4\theta$

then in expression above: $A_{mn}=0 \quad \forall m,n$

$B_{4n} \neq 0, \quad B_{mn}=0$ if $m \neq 4$

Hence

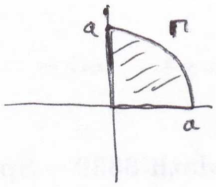
$$\boxed{u(r,\theta,t) = \sum_{n=1}^{\infty} B_{4n} J_4(\sqrt{\lambda_{4n}}\, r)\sin 4\theta}$$

$$c\sqrt{\lambda_{4n}}\, B_{4n} = \frac{\iint \alpha(r) J_4(\sqrt{\lambda_{4n}}\, r)\sin 4\theta \; r\,dr\,d\theta}{\iint (J_4(\sqrt{\lambda_{4n}}\, r)\sin 4\theta)^2 \; r\,dr\,d\theta}$$

$$\boxed{B_{4n} = \frac{1}{c\sqrt{\lambda_{4n}}}\,\frac{\iint \alpha(r) J_4(\sqrt{\lambda_{4n}}\, r)\sin^2 4\theta \; r\,dr\,d\theta}{\iint (J_4(\sqrt{\lambda_{4n}}\, r)\sin 4\theta)^2 \; r\,dr\,d\theta}}$$

$$= \boxed{\frac{1}{c\sqrt{\lambda_{4n}}}\,\frac{\int_0^a J_4(\sqrt{\lambda_{4n}}\, r)\,\alpha(r)\, r\,dr}{\int_0^a (J_4(\sqrt{\lambda_{4n}}\, r))^2\, r\,dr}}$$

(3) Problem $u_{tt} = c^2 \nabla^2 u$ $\quad 0 < r < a$, $\quad 0 < \theta < \frac{\pi}{2}$



B.C. $u(r,\theta,t) = 0$ on boundary Γ

I.C. $u(r,\theta,0) = g(r,\theta)$, $u_t(r,\theta,0) = 0$

WRITE $u(r,\theta,t) = f(r)\, g(\theta)\, h(t)$

$\hookrightarrow$ $\dfrac{d^2h}{dt^2} = -\lambda c^2 h$

$\dfrac{d^2 g}{d\theta^2} = -\mu g$

$r\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0$

From t-eq. $h(t) = a_1 \cos c\sqrt{\lambda}t + a_2 \sin c\sqrt{\lambda}t$

From θ-eq. $g(\theta) = K_1 \cos\sqrt{\mu}\theta + K_2 \sin\sqrt{\mu}\theta$

By B.C. $\underline{g(0) = K_1 = 0}$

$g(\frac{\pi}{2}) = K_2 \sin\sqrt{\mu}\frac{\pi}{2} = 0 \Rightarrow \sqrt{\mu}\frac{\pi}{2} = m\pi$ $\quad m = 0,1,2..$

$\Rightarrow \underline{\mu_m = (2m)^2}$

Hence NEED TO solve prob

$r\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) + (\lambda r^2 - (2m)^2) f = 0$

This is a Bessel's Diff. eq. with solution

$f(r) = c_1 J_{2m}(\sqrt{\lambda}r) + c_2 Y_{2m}(\sqrt{\lambda}r)$

Since $|f(0)| < \infty$, then $c_2 = 0$

Since $f(a) = 0$, then $J_{2m}(\sqrt{\lambda}a) = 0 \Rightarrow \sqrt{\lambda_{mn}}\,a = z_{2m,n}$ ← ZEROS of J_{2m}

$\boxed{\lambda_{mn} = \left(\frac{z_{2m,n}}{a}\right)^2}$

(a) Hence: $\boxed{\text{eigenvalues} : c\sqrt{\lambda_{mn}}}$

(b) Combining observations above; general solution is

$$u(r,\theta,t) = \boxed{\sum_m \sum_n A_{mn} J_{2m}(\sqrt{\lambda_{mn}}r)\sin(2m\theta)\cos c\sqrt{\lambda_{mn}}t + B_{mn} J_{2m}(\sqrt{\lambda_{mn}}r)\sin(2m\theta)\sin c\sqrt{\lambda_{mn}}t}$$

SINCE $u_t(r,\theta,0)$, then $\underline{B_{mn} = 0}$ $\forall m, n$

$$A_{mn} = \frac{\int_0^{\pi/2}\int_0^a J_{2m}(\sqrt{\lambda_{mn}}r)\sin 2m\theta\; g(r,\theta)\, r\,dr\,d\theta}{\int_0^{\pi/2}\int_0^a (J_{2m}(\sqrt{\lambda_{mn}}r)\sin 2m\theta)^2\, r\,dr\,d\theta}$$