2.5.1 Solve $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on $(0,L)\times(0,H)$

(a) B.C. $u_x(0,y) = u_x(L,y) = 0$ $u(x,0)=0$, $u(x,H) = f(x)$

SEPARATION of VARIABLES: $u(x,y) = h(x)\varphi(y)$ $\Rightarrow$ $h'(0)=h'(L)=0$

$\varphi(0)=0$ $u(x,H)=f(x)$

$$\frac{d^2 h}{dx^2} = -\lambda h \quad \text{with } h'(0)=h'(L)=0$$

$$\frac{d^2\varphi}{dy^2} = \lambda\varphi \quad \text{with } \varphi(0)=0$$

- Solution of x-dependent equation is $h(x) = \alpha_1\cos\sqrt{\lambda}x + \alpha_2\sin\sqrt{\lambda}x$

$h'(x) = -\alpha_1\sqrt{\lambda}\sin\sqrt{\lambda}x + \alpha_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$h'(0) = \sqrt{\lambda}\alpha_2 = 0$ $h'(L) = -\alpha_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0$

$\Rightarrow \boxed{\lambda_n = \frac{n\pi}{L}}$ $n=0,1,2\ldots$

$$\boxed{\varphi_n(x) = \cos\frac{n\pi}{L}x}$$

- solution of y-dependent equation:

$$\frac{d^2\varphi}{dy^2} = \left(\frac{n\pi}{L}\right)^2\varphi \quad \text{with } \varphi(0)=0$$

$n>0$ $\varphi(y) = c_1\cosh\frac{n\pi}{L}y + c_2\sinh\frac{n\pi}{L}y$

$\varphi(0) = c_1 = 0 \Rightarrow \varphi_n(y) = \sinh\frac{n\pi}{L}y$

$n=0$ $\varphi(y) = k_1 y + k_2$

$\varphi(0) = k_2 = 0 \Rightarrow \varphi_0(y) = y$

In conclusion:

$$\boxed{u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi y}{L}\right)}$$

$$u(x,H) = A_0 H + \sum_{n=1}^{\infty}\left(A_n\sinh\frac{n\pi H}{L}\right)\cos\frac{n\pi x}{L} = f(x) \quad 0\le x\le L$$

Hence $HA_0 = \frac{1}{L}\int_0^L f(x)\,dx$

$$A_n\sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$\Rightarrow \boxed{A_0 = \frac{1}{HL}\int_0^L f(x)\,dx}$$

$$\boxed{A_n = \frac{2}{L\sinh\frac{n\pi H}{L}}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx} \quad n\ge 1$$

(b) B.C. $u_x(0,y) = g(y)$, $u_x(L,y) = 0$, $u(x,0) = u(x,H) = 0$

Set $u(x,y) = h(x)\varphi(y)$ $\rightarrow$ $h'(L) = 0$, $\varphi(0) = \varphi(H) = 0$

$u_x(0,y) = g(y)$

Need to solve:

$$\frac{d^2h}{dx^2} = \lambda h \qquad h'(L) = 0$$

$$\frac{d^2\varphi}{dy^2} = -\lambda\varphi \qquad \varphi(0) = \varphi(H) = 0$$

- y-dep. eq.

$$\varphi(y) = c_1 \cos\sqrt{\lambda}y + c_2 \sin\sqrt{\lambda}y$$

$$\varphi(0) = c_1 = 0, \qquad \varphi(H) = c_2 \sin\sqrt{\lambda}H = 0 \Rightarrow \lambda_n = \left(\frac{n\pi}{H}\right)^2 \quad n = 1,2,3,\ldots$$

$$\varphi_n(y) = \sin\frac{n\pi}{H}y$$

- x-dep eq.

$$\frac{d^2h}{dx^2} = \left(\frac{n\pi}{H}\right)^2 h \qquad h'(L) = 0$$

Based on argument explained in class it must be $h(x) = c\cosh\left(\frac{n\pi}{H}(x-L)\right)$

EXPLANATION: $h(x) = k_1 \exp\left(\frac{n\pi x}{H}\right) + k_2 \exp\left(-\frac{n\pi x}{H}\right)$

$$h'(L) = \frac{n\pi}{H}\left(k_1 \exp\left(\frac{n\pi x}{H}\right) - k_2 \exp\left(-\frac{n\pi x}{H}\right)\right) = 0 \Rightarrow \frac{k_1}{k_2} = \exp\left(-\frac{2n\pi x}{H}\right)$$

choose $k_1 = \frac{c}{2}\exp\left(-\frac{n\pi L}{H}\right)$, $k_2 = \frac{c}{2}\exp\left(\frac{n\pi L}{H}\right)$

Hence $h(x) = c\cosh\left(\frac{n\pi}{H}(x-L)\right)$

In conclusion:

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi}{H}(x-L)\right)\sin\frac{n\pi}{H}y}$$

To determine A_n:

$$u_x(0,y) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{H}\sinh\left(-\frac{n\pi L}{H}\right)\sin\frac{n\pi}{H}y = g(y)$$

Hence:

$$A_n \frac{n\pi}{H}\sinh\left(-\frac{n\pi L}{H}\right) = \frac{2}{H}\int_0^H g(y)\sin\frac{n\pi}{H}y\,dy$$

$$\boxed{A_n = -\frac{2}{n\pi\sinh\left(\frac{n\pi L}{H}\right)}\int_0^H g(y)\sin\frac{n\pi y}{H}\,dy}$$

(c) B.C. $u_x(0,y)=0$, $u(L,y)=g(y)$, $u(x,0)=u(x,L)=0$

Set $u(x,y)=h(x)\varphi(y)$ $\rightarrow$ $h'(0)=0$, $\varphi(0)=\varphi(H)=0$ $u(L,y)=g(y)$

$$\frac{d^2h}{dx^2}=\lambda h \qquad h'(0)=0$$

$$\frac{d^2\varphi}{dy^2}=-\lambda\varphi \qquad \varphi(0)=\varphi(H)=0$$

- Solution of y-dependent equation.

As in part (b), $\lambda_n=\left(\frac{n\pi}{H}\right)^2$, $\varphi_n(y)=\sin\frac{n\pi y}{H}$, $n=1,2,3\ldots$

- Solution of x-dependent equation

$$\frac{d^2h}{dx^2}=\left(\frac{n\pi}{H}\right)^2 h \qquad h'(0)=0$$

$$h(x)=c_1\cosh\frac{n\pi}{H}x+c_2\sinh\frac{n\pi}{H}x$$

$$h'(x)=c_1\frac{n\pi}{H}\sinh\frac{n\pi x}{H}+c_2\frac{n\pi}{H}\cosh\frac{n\pi x}{H}$$

$$h'(0)=+c_2\frac{n\pi}{H}=0$$

Hence $h(x)=\cosh\frac{n\pi x}{H}$

We have: $$\boxed{u(x,y)=\sum_{n=1}^{\infty}A_n\cosh\frac{n\pi x}{H}\sin\frac{n\pi y}{H}}$$

$$u(L,y)=\sum_{n=1}^{\infty}A_n\cosh\frac{n\pi L}{H}\sin\frac{n\pi y}{H}=g(y)$$

Hence $$A_n\cosh\frac{n\pi L}{H}=\frac{2}{H}\int_0^H g(y)\sin\frac{n\pi y}{H}\,dy$$

$$\boxed{A_n=\frac{2}{H\cosh\frac{n\pi L}{H}}\int_0^H g(y)\sin\frac{n\pi y}{H}\,dy}$$