2.5.5 $\quad \nabla^2 u = 0 \qquad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 1$

$u(r,\theta) = \varphi(\theta) g(r)$

$\hookrightarrow \quad \frac{d^2\varphi}{d\theta^2} + \lambda\varphi = 0$

$r^2 \frac{d^2 g}{dr^2} + r\frac{dg}{dr} - \lambda g = 0$

(a) $\quad \varphi'(0) = 0, \quad \varphi(\frac{\pi}{2}) = 0, \quad u(1,\theta) = f(\theta)$

$\varphi(\theta) = c_1 \cos\sqrt{\lambda}\theta + c_2 \sin\sqrt{\lambda}\theta$

$\varphi'(\theta) = \sqrt{\lambda}(-c_1 \sin\sqrt{\lambda}\theta + c_2\cos\sqrt{\lambda}\theta)$

$\varphi'(0) = \sqrt{\lambda}c_2 = 0$

$\varphi(\frac{\pi}{2}) = c_1 \cos\sqrt{\lambda}\frac{\pi}{2} = 0 \implies \sqrt{\lambda_n}\frac{\pi}{2} = (2n-1)\frac{\pi}{2} \implies \lambda_n = (2n-1)^2, \quad n = 1,2,\ldots$

$r^2\frac{d^2 g}{dr^2} + r\frac{dg}{dr} - (2n-1)^2 g = 0 \implies g(r): \ r^{2n-1}, \ r^{-(2n-1)} \qquad n = 1,2..$

Since $|u(0,\theta)| < \infty$, we exclude soluh $r^{-(2n-1)}$

Hence $\quad u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \cos(2n-1)\theta$

$u(1,\theta) = \sum_{n=1}^{\infty} A_n \cos(2n-1)\theta = f(\theta) \implies A_n = \frac{4}{\pi}\int_0^{\pi/4} f(\theta)\cos(2n-1)\theta\, d\theta$

(b) $\quad \varphi'(0) = 0, \quad \varphi'(\frac{\pi}{2}) = 0, \quad u(1,\theta) = f(\theta)$

$\varphi(\theta) = c_1\cos\sqrt{\lambda}\theta + c_2\sin\sqrt{\lambda}\theta$

$\varphi'(0) = \sqrt{\lambda}c_2 = 0, \quad \varphi'(\frac{\pi}{2}) = -\sqrt{\lambda}c_1\sin\sqrt{\lambda}\frac{\pi}{2} = 0 \implies \lambda_n = (2n)^2 \quad n = 0,1,2..$

$\varphi_n(\theta) = \cos 2n\theta$

$r^2\frac{d^2 g}{dr^2} + r\frac{dg}{dr} - n^2 g = 0 \implies g(r): \ r^{2n}, \ r^{-2n} \quad \text{if } n > 0$

$g(r): \ 1, \ \ln r \quad \text{if } n = 0$

As above, since $|u(0,\theta)| < \infty$, we exclude $r^{-2n}$, $\ln r$

Hence $\quad u(r,\theta) = \sum_{n=0}^{\infty} A_n \cos 2n\theta \ r^{2n}$

$u(1,\theta) = \sum_{n=0}^{\infty} A_n \cos 2n\theta = f(\theta) \implies A_n = \frac{4}{\pi}\int_0^{\pi/4} f(\theta)\cos 2n\theta$

(c) $\varphi(0) = \varphi(\frac{\pi}{2}) = 0$, $\quad u_r(1,\theta) = 0$

$$\varphi(\theta) = c_1 \cos\sqrt{\lambda}\,\theta + c_2 \sin\sqrt{\lambda}\,\theta$$

$\varphi(0) = c_1 = 0 \qquad \varphi(\frac{\pi}{2}) = c_2 \sin\sqrt{\lambda}\,\frac{\pi}{2} = 0 \qquad \Rightarrow \qquad \lambda_n = (2n)^2 \qquad n = 1, 2, \ldots$

$$\varphi_n(\theta) = \sin 2n\theta$$

$$r^2 \frac{d^2 g}{dr^2} + r \frac{dg}{dr} - (2n)^2 g = 0$$

solutions $\quad g(r): \; r^{2n}, \; r^{-2n} \qquad$ As above, we exclude sol. $r^{-2n}$

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n \sin 2n\theta \; r^{2n}$$

$$u_r(r,\theta) = \sum_{n=1}^{\infty} 2n B_n \sin 2n\theta \; r^{2n-1}, \qquad u_r(1,\theta) = \sum_{n=1}^{\infty} 2n B_n \sin 2n\theta = f(\theta)$$

Hence $\quad B_n = \dfrac{4}{2n\pi} \displaystyle\int_0^{\pi/4} f(\theta) \sin 2n\theta \, d\theta$

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2.5.8 $\quad \nabla^2 u = 0 \qquad a < r < b \qquad u(a,\theta) = f(\theta), \quad u(b,\theta) = g(\theta)$

$\dfrac{d^2\varphi}{d\theta^2} + \lambda\varphi = 0 \qquad \varphi(0) = \varphi(2\pi), \quad \varphi'(0) = \varphi'(2\pi)$

$\qquad \lambda_n = n^2, \qquad n = 0, 1, 2 \ldots$

$\hookrightarrow \quad \varphi(\theta) = c_1 \cos n\theta + c_2 \sin n\theta$

$r^2 \dfrac{d^2 g}{dr^2} + r\dfrac{dg}{dr} - n^2 g = 0 \quad \rightarrow \quad g(r): \; r^n, \; r^{-n} \qquad n > 0$

$\qquad\qquad g(r): \; 1, \; \ln r \qquad n = 0$

$$u(r,\theta) = \sum_{n=1}^{\infty} d_n \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \cdot \begin{Bmatrix} r^n \\ r^{-n} \end{Bmatrix} + c \begin{Bmatrix} 1 \\ \ln r \end{Bmatrix}$$

To satisfy B.C. it is convenient to introduce radial basis function.

$$c_1\left(\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right), \; c_2\left(\left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n\right) \qquad \text{if } n > 0$$

$$c_1 \ln\frac{r}{a}, \quad c_2 \ln\frac{r}{b} \qquad \text{if } n = 0$$

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n \cos n\theta \left[\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right] + B_n \cos n\theta \left[\left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n\right] + C_n \sin n\theta \left[\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right] +$$

$$+ D_n \sin n\theta \left[\left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n\right] + A_0 \ln\frac{r}{a} + B_0 \ln\frac{r}{b}$$

$$u(a,\theta) = \sum_{n=1}^{\infty} B_n \cos n\theta \left[\left(\frac{a}{b}\right)^n - \left(\frac{b}{a}\right)^n\right] + D_n \sin n\theta \left[\left(\frac{a}{b}\right)^n - \left(\frac{b}{a}\right)^n\right] + B_0 \ln\frac{a}{b} = f(\theta)$$

$$u(b,\theta) = \sum_{n=1}^{\infty} A_n \cos n\theta \left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right] + C_n \sin n\theta \left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right] + A_0 \ln\frac{b}{a} = g(\theta)$$

$$A_n = \frac{1}{\pi\left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right]} \int_0^{2\pi} g(\theta) \cos n\theta \, d\theta, \quad B_n = \frac{1}{\pi\left[\left(\frac{a}{b}\right)^n - \left(\frac{b}{a}\right)^n\right]} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \quad A_0 = \frac{1}{2\pi \ln\frac{b}{a}} \int_0^{2\pi} g(\theta)\, d\theta$$

$$C_n = \frac{1}{\pi\left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right]} \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta, \quad D_n = \frac{1}{\pi\left[\left(\frac{a}{b}\right)^n - \left(\frac{b}{a}\right)^n\right]} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta, \quad B_0 = \frac{1}{2\pi \ln\frac{a}{b}} \int_0^{2\pi} f(\theta)\, d\theta$$