Math 3363 – Spring 2016 Name:

Test #1

Please, write clearly and justify all your steps, to get proper credit for your work. You can cite general results from the book, but no examples or exercises.

(1)[8 Pts] (a) Solve the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with

$$\text{(B.C.)} \qquad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0,$$

$$\text{(I.C.)} \qquad u(x,0) = 2 - \cos\frac{3\pi x}{L}.$$

(b) Find the solution of the problem above if the boundary condition is changed to

$$\text{(B.C.)} \qquad u(-L,t) = u(L,t), \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t).$$

(c) Compute the steady state solutions for the problems in (a-b).

(2)[8 Pts] For the following functions, state whether or not the corresponding Fourier series in the interval $[-1,1]$ converges to the function in $[-1,1]$. If it does not, indicate where it does not converge.

(a) $f(x) = 1 - x^2$

(b) $f(x) = x - x^2$

(c) $f(x) = \cos(2\pi x)$

(d) $f(x) = \sin x$

(3)[8 Pts] Compute the Fourier series of f

$$f(x) = \begin{cases} 0 & |x| \le L/2 \\ 1 & |x| > L/2 \end{cases}$$

valid in $[-L,L]$ and discuss its convergence, that is, indicate for which values of $x \in [-L,L]$ the Fouries series of f converges to f, where it does not and which value it takes at those points.

TEST #1 $\boxed{\text{SOLUTION}}$

(1)(a) GENERAL SOLUTION IS

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t}$$

For the given I.C. $A_0 = 2$, $A_3 = -1$, $A_n = 0 \quad \forall n \neq 0,3$

Hence $\boxed{u_p(x,t) = 2 - \cos\frac{3\pi x}{L}\exp\left(-\left(\frac{3\pi}{L}\right)^2 \kappa t\right)}$

(b) GENERAL SOLUTION IS

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t} + B_n \sin\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t}$$

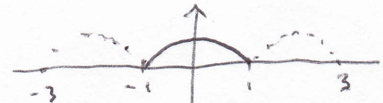
For the given I.C. $A_0 = 2$, $A_3 = -1$, $A_n = 0 \quad \forall n \neq 0,3$ $\quad B_n=0$, for all n

$u_p(x,t)$ is $\boxed{\text{SAME as PART a}}$

(c) $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} 2 - \cos\frac{3\pi x}{L}\exp\left(-\left(\frac{3\pi}{L}\right)^2 \kappa t\right) = \boxed{2}$

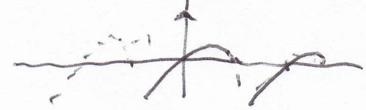
(2) (a) PERIODIC EXTENSION IS CONTINUOUS EVERYWHERE

F-SERIES CONVERGES $\boxed{\forall x \in [-1,1]}$



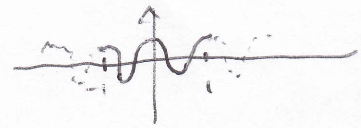
(b) PERIODIC EXTENSION IS discontinuous at $x = \pm 1$

F-SERIES converges to f $\boxed{\text{except } x = \pm 1}$



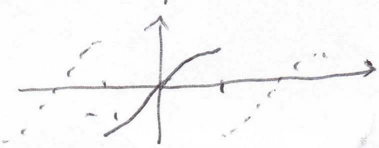
(c) PERIODIC EXTENSION IS CONTINUOUS everywhere

F-series converges to f $\boxed{\forall x \in [-1,1]}$



(d) PERIODIC EXTENSION IS discontinuous at $x = \pm 1$

F-series converges to f $\boxed{\text{except } x = \pm 1}$



(3) f IS EVEN. Hence $B_n = 0 \quad \forall n$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_{\frac{L}{2}}^{L} dx = \frac{1}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{\frac{L}{2}}^{L}\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Bigg|_{\frac{L}{2}}^{L} =$$

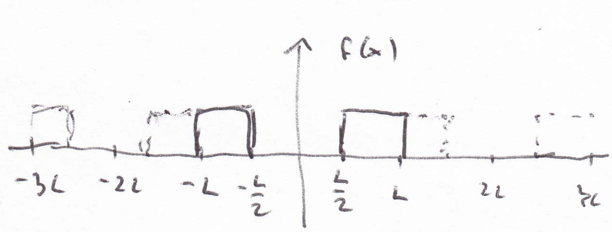
$$= \frac{2}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

NOTE: $a_{2m} = -\frac{1}{m\pi}\sin m\pi = 0$

$a_{2m-1} = \frac{-2}{(2m-1)\pi}\sin\frac{(2m-1)\pi}{2} = \frac{2}{(2m-1)\pi}(-1)^m$

Hence F-series of f:

$$\boxed{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{-2}{n\pi}\right)\sin\frac{n\pi}{2}\cos\frac{n\pi x}{L}}$$



f is piecewise smooth; its periodic extension has jumps at $x = \pm\frac{L}{2}$

$\boxed{\text{F-series converges everywhere in } [-L,L] \text{ except } x = \pm\frac{L}{2}}$