Math 3363 – Spring 2016 Name:

Test #2

Please, write clearly and justify all your steps, to get proper credit for your work. You can cite general results from the book, but no examples or exercises.

(1)[5Pts] Compute the Fourier sine series of f

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

valid in $[0, 1]$ and discuss its convergence, that is, indicate for which values of $x \in [0, 1]$ the Fourier sine series of f converges to f, where it does not and which value it takes at those points. Sketch the Fourier sine series of f.

(2)[3Pts] Consider a damped vibrating string that satisfies the equation

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}, \quad 0 < x < L,$$

Apply the method of separation of variables to derive two ordinary differential equations with respect to time and space variables.

(3)[8Pts] Consider a vibrating string of uniform density and tension, with length $L = 1$ and fixed ends. We found that vertical displacement is modelled by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,$$

with B.C. $u(0, t) = 0$, $u(1, t) = 0$. Compute and write explicitly the particular solution $u_p(x, t)$ for the given I.C. below:

(a) $u(x, 0) = 2 \sin 3\pi x, \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 < x < 1.$

(b) $u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = v_0, \quad 0 < x < 1.$

HINT: Use the calculation from the textbook (p.95-96) showing that the Fourier sine series $\sum_{n=1}^{\infty} b_n \sin n\pi x$ of the function $f(x) = 1$, for $0 \leq x \leq 1$, has coefficients $b_n = \begin{cases} 0 & \text{if } n \text{ even,} \\ 4/(n\pi) & \text{if } n \text{ odd.} \end{cases}$

① Sine series is $\sum_{n=1}^{\infty} a_n \sin n\pi x$,

where $a_n = 2\int_0^1 f(x)\sin n\pi x\,dx = 2\left[\int_0^{1/2} x \sin n\pi x\,dx + \int_{1/2}^1 (1-x)\sin n\pi x\,dx\right]$

$$= 2\left[-\frac{x}{n\pi}\cos n\pi x\Big|_0^{1/2} + \int_0^{1/2}\frac{1}{n\pi}\cos n\pi x\,dx + \int_{1/2}^1 \sin n\pi x - \int_{1/2}^1 x\sin n\pi x\,dx\right]$$

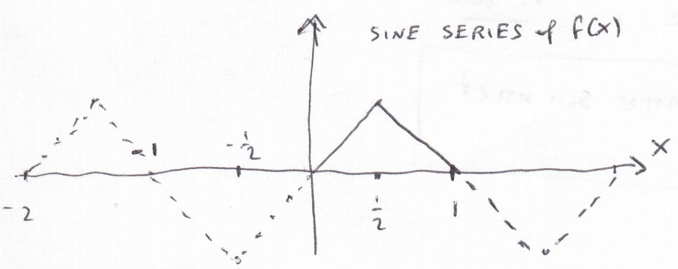
$$= 2\left[-\frac{1}{2\pi n}\cos\frac{n\pi}{2} + \frac{1}{(n\pi)^2}\sin n\pi x\Big|_0^{1/2} - \frac{1}{n\pi}\cos n\pi x\Big|_{1/2}^1 + \frac{x}{n\pi}\cos n\pi x\Big|_{1/2}^1 - \int_{1/2}^1 \frac{1}{n\pi}\cos n\pi x\,dx\right]$$

$$= 2\left[\cancel{-\frac{1}{2\pi n}\cos\frac{n\pi}{2}} + \frac{1}{(n\pi)^2}\sin\frac{n\pi}{2} - \cancel{\frac{1}{n\pi}\cos n\pi} + \cancel{\frac{1}{n\pi}\cos\frac{n\pi}{2}} + \cancel{\frac{1}{n\pi}\cos n\pi} + \right.$$

$$\left. \cancel{-\frac{1}{2\pi n}\cos\frac{n\pi}{2}} - \frac{1}{(n\pi)^2}\sin n\pi x\Big|_{1/2}^1\right]$$

$$= 2\left[\frac{1}{(n\pi)^2}\sin\frac{n\pi}{2} - \frac{1}{(n\pi)^2}\sin n\pi + \frac{1}{(n\pi)^2}\sin\frac{n\pi}{2}\right] =$$

$$= \frac{2}{(n\pi)^2}\sin\frac{n\pi}{2}$$



The periodization of the SINE SERIES is CONTINUOUS everywhere.

The SINE SERIES CONVERGES to $f(x)$ for all $x \in [0,1]$.

② Suppose $u(x,t) = \varphi(x)\,h(t)$

Into PDE:

$$\rho\,\varphi(x)\frac{d^2h}{dt^2} = T\,h(t)\frac{d^2\varphi}{dx^2} - \beta\,\varphi(x)\frac{dh}{dt}$$

Divide both sides by $\varphi(x)h(t)$.

Then

$$\rho\frac{1}{h}\frac{d^2h}{dt^2} = T\frac{1}{\varphi}\frac{d^2\varphi}{dx^2} - \beta\frac{1}{h}\frac{dh}{dt}$$

Hence

$$\frac{\rho}{T}\frac{1}{h}\frac{d^2h}{dt^2} + \frac{\beta}{T}\frac{1}{h}\frac{dh}{dt} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda$$

WE OBTAIN

$$\frac{\rho}{T}\frac{d^2h}{dt^2} + \frac{\beta}{T}\frac{dh}{dt} = -\lambda h \quad \text{or} \quad \boxed{\frac{d^2h}{dt^2} + \frac{\beta}{\rho}\frac{dh}{dt} + \lambda\frac{T}{\rho}h = 0}$$

$$\frac{d^2\varphi}{dx^2} = -\lambda\varphi \quad \text{or} \quad \boxed{\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0}$$

③ General solution of the boundary-value problem is:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin n\pi x \cos n\pi ct + B_n \sin n\pi x \sin n\pi ct \right)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x, \qquad \cancel{A_n = \frac{2}{L}\int_0^L \sin}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n\, n\pi c \sin n\pi x$$

(a) $u(x,0) = 2 \sin 3\pi x, \quad u_t(x,0) = 0$

Hence $B_n = 0 \;\forall n, \quad A_3 = 2, \quad A_n = 0 \;\forall n \neq 3$

$$\boxed{u_p(x,t) = 2 \sin 3\pi x \cos 3\pi ct}$$

(b) $u(x,0) = 0, \quad u_t(x,0) = v_0$

Hence $A_n = 0 \;\forall n, \quad \sum_{n=1}^{\infty} (B_n\, n\pi c) \sin n\pi x = v_0$

Since SINE SERIE of 1 has coeff.s $b_n = \frac{4}{n\pi}$, $\underline{n \text{ odd}}$

then $B_n\, n\pi c = \frac{4 v_0}{n\pi}$, $\underline{n \text{ odd}}$

Hence $B_n = \frac{4 v_0}{c (n\pi)^2}$, $\underline{n \text{ odd}}$

$$\boxed{u_p(x,t) = \sum_{n \text{ odd}} \frac{4 v_0}{c(n\pi)^2} \sin n\pi x \sin n\pi ct}$$