Math 3363 – Spring 2016 Name:

Test #3

Please, write clearly and justify all your steps, to get proper credit for your work. You can cite general results from the book, but no examples or exercises.

(1) [8 Pts.] Solve the Laplace's equation $\nabla^2 u = 0$ inside the rectangle $0 \le x \le L$, $0 \le y \le H$ with the following boundary conditions

$$u(0, y) = f(y),\ u(L, y) = 0,\ \frac{\partial u}{\partial y}(x, 0) = 0,\ \frac{\partial u}{\partial y}(x, H) = 0.$$

HINT: Use separation of variables $u(x, y) = h(x)\phi(y)$ to obtain

$$\frac{1}{h}\frac{d^2 h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2},$$

as done in class. You can then refer to general results for deriving the solution of Laplace's equation under this specific boundary conditions.

(2) [8 Pts.] Solve the Laplace's equation $\nabla^2 u = 0$ inside the semicircle $0 < r < a$, $0 < \theta < \pi$, subject to the boundary condition: $u(a, \theta) = f(\theta)$ for $0 < \theta < \pi$ and, on the diameter, $\frac{\partial u}{\partial \theta}(r, 0) = \frac{\partial u}{\partial \theta}(r, \pi) = 0$.
HINT: Use separation of variables in polar coordinates $u(r, \theta) = \phi(\theta)g(r)$ to obtain

$$\frac{r}{g}\frac{d}{dr}(r\frac{dg}{dr}) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$$

as done in class. You can then refer to general results for deriving the solution of Laplace's equation under this specific boundary conditions. Note that, as in similar cases, you will assume $|g(0)| < \infty$.

(3) [Extra credit. 2Pts.] Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u, \quad 0 < r < 1,\ t > 0,$$

in the circularly symmetric case $u = u(r, t)$ subject to $u(1, t) = 0$ with the initial condition $u(r, 0) = 0$, $\frac{\partial u}{\partial t}(r, 0) = 1$, $0 < r < 1$. To solve this problem, you can refer to the general solution found in class or in textbook (see eq. (7.7.64)) which was derived under same boundary condition.

(1) $\nabla^2 u = 0 \quad , \quad (x,y) \in [0,L] \times [0,H]$

$+ \ u(0,y) = f(y)$

Writing $u(x,y) = h(x)\,\varphi(y)$ we derive:

$$\frac{d^2 h}{dx^2} = \lambda h \ , \quad 0<x<L \ , \quad \text{with } h(L)=0$$

$$\frac{d^2 \varphi}{dy^2} = -\lambda \varphi \ , \quad 0<y<H \ , \quad \text{with } \varphi'(0) = \varphi'(H) = 0$$

y-dep. equat.: $\varphi(y) = c_1 \cos(\sqrt{\lambda}\, y) + c_2 \sin(\sqrt{\lambda}\, y)$

$\varphi'(y) = \sqrt{\lambda}\left(-c_1 \sin(\sqrt{\lambda}\, y) + c_2 \cos(\sqrt{\lambda}\, y)\right)$

$\varphi'(0) = \sqrt{\lambda}\, c_2 = 0 \Rightarrow c_2 = 0$

$\varphi'(H) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} H) = 0$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{H}\right)^2 , \quad n = 0,1,2 \ldots$

Hence $\varphi_n(y) = \cos \frac{n\pi}{H} y$

x-dep. equat.: $\frac{d^2 h}{dx^2} = \left(\frac{n\pi}{H}\right)^2 h \quad$ with $h(L) = 0$

$h(x) = \alpha_1 \cosh\left(\frac{n\pi}{H}(x-L)\right) + \alpha_2 \sinh\left(\frac{n\pi}{H}(x-L)\right)$

$h(L) = \alpha_1 = 0 \quad , \quad \alpha_2 \neq 0 \quad$ if $n \neq 0$

~~Hence~~ if $n=0$, $h(x) = \beta_1 x + \beta_2 \qquad h(L) = \beta_1 L + \beta_2 = 0 \Rightarrow \beta_2 = -\beta_1 L$

$\Rightarrow h(x) = c\,(x-L)$

Hence:
$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{H}(x-L)\right) \cos\left(\frac{n\pi}{H} y\right) + A_0 (x-L)}$$

$$u(0,y) = \sum_{n=1}^{\infty} -A_n \sinh\left(\frac{n\pi L}{H}\right) \cos\left(\frac{n\pi y}{H}\right) - A_0 L = f(y)$$

Hence
$$\boxed{A_n = -\frac{2}{H \sinh\left(\frac{n\pi L}{H}\right)} \int_0^H f(y) \cos\left(\frac{n\pi y}{H}\right) dy \ , \qquad A_0 = -\frac{1}{LH} \int_0^H f(y)\, dy}$$

(3) By (7.7.64) gen sol. is

$$u(r,t) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}\, r) \cos(c\sqrt{\lambda_n}\, t) + \boxed{\sum_{n=1}^{\infty} b_n J_0(\sqrt{\lambda_n}\, r) \sin(c\sqrt{\lambda_n}\, t)}$$

$u(r,0) = 0 \Rightarrow \boxed{a_n = 0} , \qquad u_t(r,0) = \sum_{n=1} c\sqrt{\lambda_n}\, b_n J_0(\sqrt{\lambda_n}\, r) = 1 \ , \quad 0<r<1$

λ_n = n-th zero of $J_0(z)$

Hence
$$\boxed{b_n = \frac{\frac{1}{c\sqrt{\lambda_n}} \int_0^1 J_0(\sqrt{\lambda_n}\, r)\, r\, dr}{\int_0^1 J_0^2(\sqrt{\lambda_n}\, r)\, r\, dr}}$$

(2) $\nabla^2 u = 0 \qquad 0 < r < a, \qquad 0 < \vartheta < \pi$

$u(r,\vartheta) = \varphi(\vartheta)\, g(r)$

$$\frac{d^2\varphi}{d\theta^2} = -\lambda\varphi \qquad \varphi'(0) = \varphi'(\pi) = 0$$

$$+\ u(a,\theta) = f(\theta) \quad \text{f } 0 < \vartheta < \pi$$

$$r^2\frac{d^2 g}{dr^2} + r\frac{dg}{dr} + \lambda g = 0$$

θ- equatn: $\varphi(\theta) = c_1\cos(\sqrt{\lambda}\,\vartheta) + c_2\sin(\sqrt{\lambda}\,\vartheta)$

$\varphi'(\theta) = -\sqrt{\lambda}\,(c_1\sin(\sqrt{\lambda}\,\vartheta) + c_2\cos(\sqrt{\lambda}\,\theta)) \qquad \varphi'(0) = -\sqrt{\lambda}\, c_2 = 0$

$\varphi'(\pi) = -\sqrt{\lambda}\, c_1\sin(\sqrt{\lambda}\,\pi) \Rightarrow \lambda_n = n^2$

$\varphi_n(\theta) = \cos(n\,\vartheta), \qquad n = 0, 1, 2 \ldots$

r- equatn: $r^2\dfrac{d^2 g}{dr^2} + r\dfrac{dg}{dr} - n^2 g = 0$

solutions: $g(r) = \alpha_1 r^n + \alpha_2 r^{-n}$ if $n \neq 0$, $\quad g(r) = \beta_1 + \beta_2 \ln r$ if $n = 0$

Since $|g(0)| < \infty$, we exclude solutions r^{-n} and $\ln r$

Hence: $\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n \cos(n\vartheta)\, r^n}$

$$u(a,\vartheta) = \sum_{n=0}^{\infty} A_n a^n \cos(n\vartheta) = f(\theta) \qquad 0 < \vartheta < \pi$$

Hence:
$$\boxed{\begin{aligned} A_0 &= \frac{1}{\pi}\int_0^{\pi} f(\theta)\, d\vartheta \\ A_n &= \frac{2}{a^n \pi}\int_0^{\pi} f(\theta)\cos(n\vartheta)\, d\vartheta \end{aligned}}$$