order properties described in the preceding sections, but we have seen that $\sqrt{2}$ *cannot* be represented as a rational number; therefore $\sqrt{2}$ does not belong to $\mathbb{Q}$. This observation shows the necessity of an additional property to characterize the real number system. This additional property, the Completeness (or the Supremum) Property, is an essential property of $\mathbb{R}$, and we will say that $\mathbb{R}$ is a *complete ordered field*. It is this special property that permits us to define and develop the various limiting procedures that will be discussed in the chapters that follow.

There are several different ways to describe the Completeness Property. We choose to give what is probably the most efficient approach by assuming that each nonempty bounded subset of $\mathbb{R}$ has a supremum.

Suprema and Infima

We now introduce the notions of upper bound and lower bound for a set of real numbers. These ideas will be of utmost importance in later sections.

2.3.1 Definition Let S be a nonempty subset of $\mathbb{R}$.

(a) The set S is said to be **bounded above** if there exists a number $u \in \mathbb{R}$ such that $s \le u$ for all $s \in S$. Each such number u is called an **upper bound** of S.

(b) The set S is said to be **bounded below** if there exists a number $w \in \mathbb{R}$ such that $w \le s$ for all $s \in S$. Each such number w is called a **lower bound** of S.

(c) A set is said to be **bounded** if it is both bounded above and bounded below. A set is said to be **unbounded** if it is not bounded.

For example, the set $S := \{x \in \mathbb{R} : x < 2\}$ is bounded above; the number 2 and any number larger than 2 is an upper bound of S. This set has no lower bounds, so that the set is not bounded below. Thus it is unbounded (even though it is bounded above).

If a set has one upper bound, then it has infinitely many upper bounds, because if u is an upper bound of S, then the numbers $u+1, u+2, \cdots$ are also upper bounds of S. (A similar observation is valid for lower bounds.)

In the set of upper bounds of S and the set of lower bounds of S, we single out their least and greatest elements, respectively, for special attention in the following definition. (See Figure 2.3.1.)

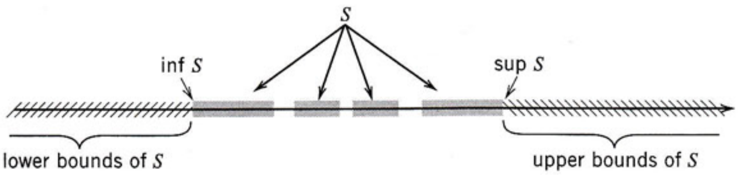

Figure 2.3.1 $\inf S$ and $\sup S$

2.3.2 Definition Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above, then a number u is said to be a **supremum** (or a **least upper bound**) of S if it satisfies the conditions:

(1) u is an upper bound of S, and

(2) if v is any upper bound of S, then $u \le v$.

(b) If S is bounded below, then a number w is said to be an **infimum** (or a **greatest lower bound**) of S if it satisfies the conditions:

(1′) w is a lower bound of S, and
(2′) if t is any lower bound of S, then $t \leq w$.

It is not difficult to see that *there can be only one supremum of a given subset S of $\mathbb{R}$.* (Then we can refer to *the* supremum of a set instead of *a* supremum.) For, suppose that u_1 and u_2 are both suprema of S. If $u_1 < u_2$, then the hypothesis that u_2 is a supremum implies that u_1 cannot be an upper bound of S. Similarly, we see that $u_2 < u_1$ is not possible. Therefore, we must have $u_1 = u_2$. A similar argument can be given to show that the infimum of a set is uniquely determined.

If the supremum or the infimum of a set S exists, we will denote them by

$$\sup S \quad \text{and} \quad \inf S.$$

We also observe that if u' is an arbitrary upper bound of a nonempty set S, then $\sup S \leq u'$. This is because $\sup S$ is the least of the upper bounds of S.

First of all, it needs to be emphasized that in order for a nonempty set S in $\mathbb{R}$ to have a supremum, it must have an upper bound. Thus, not every subset of $\mathbb{R}$ has a supremum; similarly, not every subset of $\mathbb{R}$ has an infimum. Indeed, there are four possibilities for a nonempty subset S of $\mathbb{R}$: it can

(i) have both a supremum and an infimum,
(ii) have a supremum but no infimum,
(iii) have a infimum but no supremum,
(iv) have neither a supremum nor an infimum.

We also wish to stress that in order to show that $u = \sup S$ for some nonempty subset S of $\mathbb{R}$, we need to show that *both* (1) and (2) of Definition 2.3.2(a) hold. It will be instructive to reformulate these statements. First the reader should see that the following two statements about a number u and a set S are equivalent:

(1) u is an upper bound of S,
(1′) $s \leq u$ for all $s \in S$.

Also, the following statements about an upper bound u of a set S are equivalent:

(2) if v is any upper bound of S, then $u \leq v$,
(2′) if $z < u$, then z is not an upper bound of S,
(2″) if $z < u$, then there exists $s_z \in S$ such that $z < s_z$,
(2‴) if $\varepsilon > 0$, then there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Therefore, we can state two alternate formulations for the supremum.

2.3.3 Lemma *A number u is the supremum of a nonempty subset S of $\mathbb{R}$ if and only if u satisfies the conditions:*

(1) $s \leq u$ *for all* $s \in S$,
(2) *if* $v < u$, *then there exists* $s' \in S$ *such that* $v < s'$.

We leave it to the reader to write out the details of the proof.

2.3.4 Lemma *An upper bound u of a nonempty set S in $\mathbb{R}$ is the supremum of S if and only if for every $\varepsilon > 0$ there exists an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.*

Proof. If u is an upper bound of S that satisfies the stated condition and if $v < u$, then we put $\varepsilon := u - v$. Then $\varepsilon > 0$, so there exists $s_\varepsilon \in S$ such that $v = u - \varepsilon < s_\varepsilon$. Therefore, v is not an upper bound of S, and we conclude that $u = \sup S$.

Conversely, suppose that $u = \sup S$ and let $\varepsilon > 0$. Since $u - \varepsilon < u$, then $u - \varepsilon$ is not an upper bound of S. Therefore, some element s_ε of S must be greater than $u - \varepsilon$; that is, $u - \varepsilon < s_\varepsilon$. (See Figure 2.3.2.)

Q.E.D.

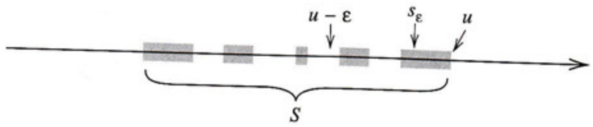

Figure 2.3.2 $u = \sup S$

It is important to realize that the supremum of a set may or may not be an element of the set. Sometimes it is and sometimes it is not, depending on the particular set. We consider a few examples.

2.3.5 Examples **(a)** If a nonempty set S_1 has a finite number of elements, then it can be shown that S_1 has a largest element u and a least element w. Then $u = \sup S_1$ and $w = \inf S_1$, and they are both members of S_1. (This is clear if S_1 has only one element, and it can be proved by induction on the number of elements in S_1; see Exercises 11 and 12.)

(b) The set $S_2 := \{x : 0 \le x \le 1\}$ clearly has 1 for an upper bound. We prove that 1 is its supremum as follows. If $v < 1$, there exists an element $s' \in S_2$ such that $v < s'$. (Name one such element s'.) Therefore v is not an upper bound of S_2 and, since v is an arbitrary number $v < 1$, we conclude that $\sup S_2 = 1$. It is similarly shown that $\inf S_2 = 0$. Note that both the supremum and the infimum of S_2 are contained in S_2.

(c) The set $S_3 := \{x : 0 < x < 1\}$ clearly has 1 for an upper bound. Using the same argument as given in (b), we see that $\sup S_3 = 1$. In this case, the set S_3 does *not* contain its supremum. Similarly, $\inf S_3 = 0$ is not contained in S_3. □

The Completeness Property of $\mathbb{R}$

It is not possible to prove on the basis of the field and order properties of $\mathbb{R}$ that were discussed in Section 2.1 that every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$. However, it is a deep and fundamental property of the real number system that this is indeed the case. We will make frequent and essential use of this property, especially in our discussion of limiting processes. The following statement concerning the existence of suprema is our final assumption about $\mathbb{R}$. Thus, we say that $\mathbb{R}$ is a *complete ordered field.*

2.3.6 The Completeness Property of $\mathbb{R}$ *Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

This property is also called the **Supremum Property** of $\mathbb{R}$. The analogous property for infima can be deduced from the Completeness Property as follows. Suppose that S is a nonempty subset of $\mathbb{R}$ that is bounded below. Then the nonempty set $\overline{S} := \{-s : s \in S\}$ is bounded above, and the Supremum Property implies that $u := \sup \overline{S}$ exists in $\mathbb{R}$. The reader should verify in detail that $-u$ is the infimum of S.

Exercises for Section 2.3

1. Let $S_1 := \{x \in \mathbb{R} : x \geq 0\}$. Show in detail that the set S_1 has lower bounds, but no upper bounds. Show that $\inf S_1 = 0$.
2. Let $S_2 = \{x \in \mathbb{R} : x > 0\}$. Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.
3. Let $S_3 = \{1/n : n \in \mathbb{N}\}$. Show that $\sup S_3 = 1$ and $\inf S_3 \geq 0$. (It will follow from the Archimedean Property in Section 2.4 that $\inf S_3 = 0$.)
4. Let $S_4 := \{1 - (-1)^n/n : n \in \mathbb{N}\}$. Find $\inf S_4$ and $\sup S_4$.
5. Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = -\sup\{-s : s \in S\}$.
6. If a set $S \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of S.
7. Let $S \subseteq \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound of S if and only if the conditions $t \in \mathbb{R}$ and $t > u$ imply that $t \notin S$.
8. Let $S \subseteq \mathbb{R}$ be nonempty. Show that if $u = \sup S$, then for every number $n \in \mathbb{N}$ the number $u - 1/n$ is not an upper bound of S, but the number $u + 1/n$ is an upper bound of S. (The converse is also true; see Exercise 2.4.3.)
9. Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Show that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.
10. Let S be a bounded set in $\mathbb{R}$ and let S_0 be a nonempty subset of S. Show that $\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$.
11. Let $S \subseteq \mathbb{R}$ and suppose that $s^* := \sup S$ belongs to S. If $u \notin S$, show that $\sup(S \cup \{u\}) = \sup\{s^*, u\}$.
12. Show that a nonempty finite set $S \subseteq \mathbb{R}$ contains its supremum. [Hint: Use Mathematical Induction and the preceding exercise.]
13. Show that the assertions (1) and (1′) before Lemma 2.3.3 are equivalent.
14. Show that the assertions (2), (2′), (2″), and (2‴) before Lemma 2.3.3 are equivalent.
15. Write out the details of the proof of Lemma 2.3.3.

Section 2.4 Applications of the Supremum Property

We will now discuss how to work with suprema and infima. We will also give some very important applications of these concepts to derive fundamental properties of $\mathbb{R}$. We begin with examples that illustrate useful techniques in applying the ideas of supremum and infimum.

2.4.1 Example **(a)** It is an important fact that taking suprema and infima of sets is compatible with the algebraic properties of $\mathbb{R}$. As an example, we present here the compatibility of taking suprema and addition.

Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any number in $\mathbb{R}$. Define the set $a + S := \{a + s : s \in S\}$. We will prove that

$$\sup(a + S) = a + \sup S.$$

If we let $u := \sup S$, then $x \le u$ for all $x \in S$, so that $a + x \le a + u$. Therefore, $a + u$ is an upper bound for the set $a + S$; consequently, we have $\sup(a + S) \le a + u$.

Now if v is *any* upper bound of the set $a + S$, then $a + x \le v$ for all $x \in S$. Consequently $x \le v - a$ for all $x \in S$, so that $v - a$ is an upper bound of S. Therefore, $u = \sup S \le v - a$, which gives us $a + u \le v$. Since v is any upper bound of $a + S$, we can replace v by $\sup(a + S)$ to get $a + u \le \sup(a + S)$.

Combining these inequalities, we conclude that

$$\sup(a + S) = a + u = a + \sup S.$$

For similar relationships between the suprema and infima of sets and the operations of addition and multiplication, see the exercises.

(b) If the suprema or infima of two sets are involved, it is often necessary to establish results in two stages, working with one set at a time. Here is an example.

Suppose that A and B are nonempty subsets of $\mathbb{R}$ that satisfy the property:

$$a \le b \qquad \text{for all } a \in A \text{ and all } b \in B.$$

We will prove that

$$\sup A \le \inf B.$$

For, given $b \in B$, we have $a \le b$ for all $a \in A$. This means that b is an upper bound of A, so that $\sup A \le b$. Next, since the last inequality holds for all $b \in B$, we see that the number $\sup A$ is a lower bound for the set B. Therefore, we conclude that $\sup A \le \inf B$. □

Functions

The idea of upper bound and lower bound is applied to functions by considering the range of a function. Given a function $f : D \to \mathbb{R}$, we say that f is **bounded above** if the set $f(D) = \{f(x) : x \in D\}$ is bounded above in $\mathbb{R}$; that is, there exists $B \in \mathbb{R}$ such that $f(x) \le B$ for all $x \in D$. Similarly, the function f is **bounded below** if the set $f(D)$ is bounded below. We say that f is **bounded** if it is bounded above and below; this is equivalent to saying that there exists $B \in \mathbb{R}$ such that $|f(x)| \le B$ for all $x \in D$.

The following example illustrates how to work with suprema and infima of functions.

2.4.2 Example Suppose that f and g are real-valued functions with common domain $D \subseteq \mathbb{R}$. We assume that f and g are bounded.

(a) If $f(x) \le g(x)$ for all $x \in D$, then $\sup f(D) \le \sup g(D)$, which is sometimes written:

$$\sup_{x \in D} f(x) \le \sup_{x \in D} g(x).$$

We first note that $f(x) \le g(x) \le \sup g(D)$, which implies that the number $\sup g(D)$ is an upper bound for $f(D)$. Therefore, $\sup f(D) \le \sup g(D)$.

(b) We note that the hypothesis $f(x) \le g(x)$ for all $x \in D$ in part (a) does not imply any relation between $\sup f(D)$ and $\inf g(D)$.

For example, if $f(x) := x^2$ and $g(x) := x$ with $D = \{x : 0 \le x \le 1\}$, then $f(x) \le g(x)$ for all $x \in D$. However, we see that $\sup f(D) = 1$ and $\inf g(D) = 0$. Since $\sup g(D) = 1$, the conclusion of (a) holds.

(c) If $f(x) \leq g(y)$ for all $x, y \in D$, then we may conclude that $\sup f(D) \leq \inf g(D)$, which we may write as:

$$\sup_{x \in D} f(x) \leq \inf_{y \in D} g(y).$$

(Note that the functions in (b) do not satisfy this hypothesis.)

The proof proceeds in two stages as in Example 2.4.1(b). The reader should write out the details of the argument. □

Further relationships between suprema and infima of functions are given in the exercises.

The Archimedean Property

Because of your familiarity with the set $\mathbb{R}$ and the customary picture of the real line, it may seem obvious that the set $\mathbb{N}$ of natural numbers is *not* bounded in $\mathbb{R}$. How can we prove this "obvious" fact? In fact, we cannot do so by using only the Algebraic and Order Properties given in Section 2.1. Indeed, we must use the Completeness Property of $\mathbb{R}$ as well as the Inductive Property of $\mathbb{N}$ (that is, if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$).

The absence of upper bounds for $\mathbb{N}$ means that given any real number x there exists a natural number n (depending on x) such that $x < n$.

2.4.3 Archimedean Property If $x \in \mathbb{R}$, *then there exists* $n_x \in \mathbb{N}$ *such that* $x < n_x$.

Proof. If the assertion is false, then $n \leq x$ for all $n \in \mathbb{N}$; therefore, x is an upper bound of $\mathbb{N}$. Therefore, by the Completeness Property, the nonempty set $\mathbb{N}$ has a supremum $u \in \mathbb{R}$. Subtracting 1 from u gives a number $u - 1$ which is smaller than the supremum u of $\mathbb{N}$. Therefore $u - 1$ is not an upper bound of $\mathbb{N}$, so there exists $m \in \mathbb{N}$ with $u - 1 < m$. Adding 1 gives $u < m + 1$, and since $m + 1 \in \mathbb{N}$, this inequality contradicts the fact that u is an upper bound of $\mathbb{N}$. Q.E.D.

2.4.4 Corollary *If* $S := \{1/n : n \in \mathbb{N}\}$, *then* $\inf S = 0$.

Proof. Since $S \neq \emptyset$ is bounded below by 0, it has an infimum and we let $w := \inf S$. It is clear that $w \geq 0$. For any $\varepsilon > 0$, the Archimedean Property implies that there exists $n \in \mathbb{N}$ such that $1/\varepsilon < n$, which implies $1/n < \varepsilon$. Therefore we have

$$0 \leq w \leq 1/n < \varepsilon.$$

But since $\varepsilon > 0$ is arbitrary, it follows from Theorem 2.1.9 that $w = 0$. Q.E.D.

2.4.5 Corollary *If* $t > 0$, *there exists* $n_t \in \mathbb{N}$ *such that* $0 < 1/n_t < t$.

Proof. Since $\inf\{1/n : n \in \mathbb{N}\} = 0$ and $t > 0$, then t is not a lower bound for the set $\{1/n : n \in \mathbb{N}\}$. Thus there exists $n_t \in \mathbb{N}$ such that $0 < 1/n_t < t$. Q.E.D.

2.4.6 Corollary *If* $y > 0$, *there exists* $n_y \in \mathbb{N}$ *such that* $n_y - 1 \leq y < n_y$.

Proof. The Archimedean Property ensures that the subset $E_y := \{m \in \mathbb{N} : y < m\}$ of $\mathbb{N}$ is not empty. By the Well-Ordering Property 1.2.1, E_y has a least element, which we denote by n_y. Then $n_y - 1$ does not belong to E_y, and hence we have $n_y - 1 \le y < n_y$. Q.E.D.

Collectively, the Corollaries 2.4.4–2.4.6 are sometimes referred to as the Archimedean Property of $\mathbb{R}$.

The Existence of $\sqrt{2}$

The importance of the Supremum Property lies in the fact that it guarantees the existence of real numbers under certain hypotheses. We shall make use of it in this way many times. At the moment, we shall illustrate this use by proving the existence of a positive real number x such that $x^2 = 2$; that is, the positive square root of 2. It was shown earlier (see Theorem 2.1.4) that such an x cannot be a rational number; thus, we will be deriving the existence of at least one irrational number.

2.4.7 Theorem *There exists a positive real number x such that $x^2 = 2$.*

Proof. Let $S := \{s \in \mathbb{R}: 0 \le s, s^2 < 2\}$. Since $1 \in S$, the set is not empty. Also, S is bounded above by 2, because if $t > 2$, then $t^2 > 4$ so that $t \notin S$. Therefore the Supremum Property implies that the set S has a supremum in $\mathbb{R}$, and we let $x := \sup S$. Note that $x > 1$.

We will prove that $x^2 = 2$ by ruling out the other two possibilities: $x^2 < 2$ and $x^2 > 2$.

First assume that $x^2 < 2$. We will show that this assumption contradicts the fact that $x = \sup S$ by finding an $n \in \mathbb{N}$ such that $x + 1/n \in S$, thus implying that x is not an upper bound for S. To see how to choose n, note that $1/n^2 \le 1/n$ so that

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \le x^2 + \frac{1}{n}(2x+1).$$

Hence if we can choose n so that

$$\frac{1}{n}(2x+1) < 2 - x^2,$$

then we get $(x + 1/n)^2 < x^2 + (2 - x^2) = 2$. By assumption we have $2 - x^2 > 0$, so that $(2 - x^2)/(2x + 1) > 0$. Hence the Archimedean Property (Corollary 2.4.5) can be used to obtain $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \frac{2 - x^2}{2x + 1}.$$

These steps can be reversed to show that for this choice of n we have $x + 1/n \in S$, which contradicts the fact that x is an upper bound of S. Therefore we cannot have $x^2 < 2$.

Now assume that $x^2 > 2$. We will show that it is then possible to find $m \in \mathbb{N}$ such that $x - 1/m$ is also an upper bound of S, contradicting the fact that $x = \sup S$. To do this, note that

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}.$$

Hence if we can choose m so that

then $(x - 1/m)^2 > x^2 - (x^2 - 2) = 2$. Now by assumption we have $x^2 - 2 > 0$, so that $(x^2 - 2)/2x > 0$. Hence, by the Archimedean Property, there exists $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \frac{x^2 - 2}{2x}.$$

These steps can be reversed to show that for this choice of m we have $(x - 1/m)^2 > 2$. Now if $s \in S$, then $s^2 < 2 < (x - 1/m)^2$, whence it follows from 2.1.13(a) that $s < x - 1/m$. This implies that $x - 1/m$ is an upper bound for S, which contradicts the fact that $x = \sup S$. Therefore we cannot have $x^2 > 2$.

Since the possibilities $x^2 < 2$ and $x^2 > 2$ have been excluded, we must have $x^2 = 2$.
Q.E.D.

By slightly modifying the preceding argument, the reader can show that if $a > 0$, then there is a unique $b > 0$ such that $b^2 = a$. We call b the **positive square root** of a and denote it by $b = \sqrt{a}$ or $b = a^{1/2}$. A slightly more complicated argument involving the binomial theorem can be formulated to establish the existence of a unique **positive *n*th root** of a, denoted by $\sqrt[n]{a}$ or $a^{1/n}$, for each $n \in \mathbb{N}$.

Remark If in the proof of Theorem 2.4.7 we replace the set S by the set of rational numbers $T := \{r \in \mathbb{Q} : 0 \le r, r^2 < 2\}$, the argument then gives the conclusion that $y := \sup T$ satisfies $y^2 = 2$. Since we have seen in Theorem 2.1.4 that y cannot be a rational number, it follows that the set T that consists of rational numbers does not have a supremum belonging to the set $\mathbb{Q}$. Thus the ordered field $\mathbb{Q}$ of rational numbers does *not* possess the Completeness Property.

Density of Rational Numbers in $\mathbb{R}$

We now know that there exists at least one irrational real number, namely $\sqrt{2}$. Actually there are "more" irrational numbers than rational numbers in the sense that the set of rational numbers is countable (as shown in Section 1.3), while the set of irrational numbers is uncountable (see Section 2.5). However, we next show that in spite of this apparent disparity, the set of rational numbers is "dense" in $\mathbb{R}$ in the sense that given any two real numbers there is a rational number between them (in fact, there are infinitely many such rational numbers).

2.4.8 The Density Theorem *If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. It is no loss of generality (why?) to assume that $x > 0$. Since $y - x > 0$, it follows from Corollary 2.4.5 that there exists $n \in \mathbb{N}$ such that $1/n < y - x$. Therefore, we have $nx + 1 < ny$. If we apply Corollary 2.4.6 to $nx > 0$, we obtain $m \in \mathbb{N}$ with $m - 1 \le nx < m$. Therefore, $m \le nx + 1 < ny$, whence $nx < m < ny$. Thus, the rational number $r := m/n$ satisfies $x < r < y$. Q.E.D.

To round out the discussion of the interlacing of rational and irrational numbers, we have the same "betweenness property" for the set of irrational numbers.

2.4.9 Corollary *If x and y are real numbers with $x < y$, then there exists an irrational number z such that $x < z < y$.*

Proof. If we apply the Density Theorem 2.4.8 to the real numbers $x/\sqrt{2}$ and $y/\sqrt{2}$, we obtain a rational number $r \neq 0$ (why?) such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

Then $z := r\sqrt{2}$ is irrational (why?) and satisfies $x < z < y$. Q.E.D.

Exercises for Section 2.4

1. Show that $\sup\{1 - 1/n : n \in \mathbb{N}\} = 1$.
2. If $S := \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.
3. Let $S \subseteq \mathbb{R}$ be nonempty. Prove that if a number u in $\mathbb{R}$ has the properties: (i) for every $n \in \mathbb{N}$ the number $u - 1/n$ is not an upper bound of S, and (ii) for every number $n \in \mathbb{N}$ the number $u + 1/n$ is an upper bound of S, then $u = \sup S$. (This is the converse of Exercise 2.3.8.)
4. Let S be a nonempty bounded set in $\mathbb{R}$.
   (a) Let $a > 0$, and let $aS := \{as : s \in S\}$. Prove that
   $$\inf(aS) = a \inf S, \qquad \sup(aS) = a \sup S.$$
   (b) Let $b < 0$ and let $bS = \{bs : s \in S\}$. Prove that
   $$\inf(bS) = b \sup S, \qquad \sup(bS) = b \inf S.$$
5. Let X be a nonempty set and let $f : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, show that Example 2.4.1(a) implies that
   $$\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}$$
   Show that we also have
   $$\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}$$
6. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.
7. Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that
   $$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$
   and that
   $$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x) + g(x) : x \in X\}.$$
   Give examples to show that each of these inequalities can be either equalities or strict inequalities.
8. Let $X = Y := \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) := 2x + y$.
   (a) For each $x \in X$, find $f(x) := \sup\{h(x, y) : y \in Y\}$; then find $\inf\{f(x) : x \in X\}$.
   (b) For each $y \in Y$, find $g(y) := \inf\{h(x, y) : x \in X\}$; then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (a).
9. Perform the computations in (a) and (b) of the preceding exercise for the function $h : X \times Y \to \mathbb{R}$ defined by
   $$h(x, y) := \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x \geq y. \end{cases}$$
10. Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be defined by
    $$f(x) := \sup\{h(x, y) : y \in Y\}, \qquad g(y) := \inf\{h(x, y) : x \in X\}.$$