HW # 10 SOLUTION

6.1 E Suppose f EVEN

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{f(x-h) - f(x)}{h} = \lim_{h\to 0} -\left(\frac{f(x-h) - f(x)}{(-h)}\right) = -f'(x)$$

Hence $f'(x)$ is ODD

Suppose f ODD

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{-f(x-h) + f(x)}{h} = \lim_{h\to 0} \frac{f(x-h) - f(x)}{-h} = f'(x)$$

Hence $f'(x)$ is EVEN

6.1 R Denote $\frac{d^n}{dx^n} f(x) = f^{(n)}(x)$

For $n = 0$ $\quad (f(x)\, g(x))^{(0)} = f(x)\, g(x)$

For n, assume $\quad (f(x)\, g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)\, g^{(n-k)}(x)$

For $n+1$,

$$(f(x)g(x))^{(n+1)} = \frac{d}{dx}(f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dx}\left[f^{k}(x)\, g^{n-k}(x)\right] =$$

$$= \sum_{k=0}^{n} \binom{n}{k}\left[f^{(k+1)}(x)\, y^{n-k}(x) + f^{(k)}(x)\, g^{n-k+1}(x)\right] =$$

$$= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)}(x)\, g^{(n-k+1)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)\, y^{(n-k+1)}(x)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{k}(x)\, y^{n-k+1}(x)$$

$\boxed{\text{USE } \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}}$

NOTE $\binom{n}{0} = \binom{n+1}{0} = 1$

$\binom{n}{n} = \binom{n+1}{n+1} = 1$

6.2 B Since $f'(x)$ is continuous on (a,b) and $f'(x_0) \neq 0$ at $x_0 \in (a,b)$, then $f'(x)$ will be $\neq 0$ on a small interval near x_0. WLOG, suppose $f'(x_0) > 0$. Then there is an INTERVAL $(x_0-\delta, x_0+\delta)$, for some $\delta > 0$, where $f'(x) > 0$. It follows, by MVT, that $f(x)$ is strictly INCREASING and, hence 1-1, on $(x_0-\delta, x_0+\delta)$

6.2 L Suppose $x < y$ and $z = tx + (1-t)y$, $t \in [0,1]$. [NOTE: $(z-y) = t(x-y)$, $(z-x) = (1-t)(y-x)$]

(a) By MVT, $\exists\, a \in [x,z]$, $b \in [z,y]$ s.t. $\frac{f(z) - f(x)}{z - x} = f'(a)$, $\frac{f(y) - f(z)}{y - z} = f'(b)$.

Since f' INCREASING, then $\frac{f(z) - f(x)}{z-x} \le \frac{f(y) - f(z)}{y - z}$

Hence $\cancel{f(z) \le f(x) + t(f(y) - f(z))} =$ $(y-z)\,(f(z) - f(x)) \le (z-x)\,(f(y) - f(z))$

Hence $(y-x)\, f(z) \le (y-z)\, f(x) + (z-x)\, f(y) = t(y-x)\, f(x) + (1-t)(y-x)\, f(y)$

Hence $f(z) \le t\, f(x) + (1-t)\, f(y)$

Hence f is convex

(b) If $f''(x_0) > 0$, then $f'(x_0)$ is INCREASING by MVT, near x_0. Hence f is convex near x_0 by (a)

(c) If $f''(x) \ge 0$ $\forall x$, then $f'(x)$ is INCREASING for all x, hence f is convex by part (a).

6.3 C

$U(f,P) = \sum_j M_j(f,P)\Delta_j$, $\quad L(f,P) = \sum_j m_j(f,P)\Delta_j$

For fixed j, $\quad M_j(f,P) - m_j(f,P) = \max_{s\in[x_{j-1},x_j]} f(x) - \min_{s\in[x_{j-1},x_j]} f(x) \le c(x^* - x^{**}) \le c(x_j - x_{j-1})$

where x^*, x^{**} are max and min of f in Δ_j

Hence $U(f,P) - L(f,P) \le \sum_j c(x_j - x_{j-1})\Delta_j \le c\,\text{mesh}(P)\sum_j (x_j - x_{j-1}) =$

$= c(b-a)\,\text{mesh}(P)$

6.3 F

Suppose f is INTEGRABLE. This means that there is a partition P s.t. $U(f,P) - L(f,P) < \varepsilon$, for any $\varepsilon > 0$.

$M_j(|f|,P) - m_j(|f|,P) = \max_{x\in[x_{j-1},x_j]} |f(x)| - \min_{x\in(\cdot)} |f(x)| = \big||f(x_j^*)| - |f(x_j^{**})|\big|$ for some $x_j^*, x_j^{**} \in [x_{j-1}, x_j]$

We have $\big||f(x_j^*)| - |f(x_j^{**})|\big| = \big||f(x_j^*)| - |f(x_j^{**})|\big| \le |f(x_j^*) - f(x_j^{**})| = f(x_j^*) - f(x_j^{**})$

we can remove abs. value since one is max, other is min.

$M_j(|f|,P) - m_j(|f|,P) \le M_j(f,P) - m_j(f,P)$.

This shows that $|f|$ is INTEGRABLE.

Hence

6.3 I

$\cancel{\int_{a+c}^{b+c} g(x)\,dx}$. Let $P = \{x_0 = a < x_1 \dots < x_n = b\}$ be a partition of $[a,b]$

and set $P_c = \{x_0 + c, x_1 + c, \dots x_n + c\}$. This is a partition of $[a+c, b+c]$

Now: $M_j(f,P) = M_j(g,P_c)$, $\quad m_j(f,P) = m_j(g,P_c)$

Then $U(f,P) = U(g,P_c)$, $\quad L(f,P) = L(g,P_c)$

It follows that f is INTEGRABLE on $[a,b]$ iff g is INTEGRABLE on $[a+c, b+c]$

This implies that $\int_a^b f = \int_{a+c}^{b+c} g$

6.4 C

By FTC, $F(x) = \int_a^x f(t)\,dt$, $\quad x \in [a,b]$ is CONTINUOUS, DIFFERENTIABLE

and $F'(x) = f(x)$. By MVT

$F(b) - F(a) = F'(c)(b-a) = f(c)(b-a)$, $\quad c \in [a,b]$.

Hence $\frac{1}{b-a}(F(b) - F(a)) = \frac{1}{b-a}\int_a^b f(t)\,dt = f(c)$