HW #1 SOLUTION

① (a) We proved that $-x \le |x| \le x$, $-y \le |y| \le y$, $\forall x, y \in \mathbb{R}$

Combining the two sets of inequalities it follows that $-(|x|+|y|) \le x+y \le |x|+|y|$

Thus $|x+y| \le |x| + |y| \quad \forall x, y \in \mathbb{R}$

(b) Let P_n : $|x_1 + \dots x_n| \le |x_1| + \dots |x_n|$ $n \ge 2$

P_2 : $|x_1 + x_2| \le |x_1| + |x_2|$ by triangle ineq.

Assume P_n holds.

P_{n+1} : $|x_1 + \dots + x_n + x_{n+1}| \le |x_1 + \dots + x_n| + |x_{n+1}|$ by triangle ineq.

$\le |x_1| + \dots |x_n| + |x_{n+1}|$ by INDUCTION step P_n

(c) Using triangle ineq: $|x| - |y| \le |x - y| + |y| - |y| = |x-y|$

$|y| - |x| \le |y - x| + |x| - |x| = |y - x|$

Hence $||x| - |y|| \le |x - y|$

② Assume $|x-a| < \varepsilon$. then $-\varepsilon < x - a < \varepsilon$

then $a - \varepsilon < x < a + \varepsilon$

Assume $a - \varepsilon < x < a + \varepsilon$. then $-\varepsilon < x - a < \varepsilon$

then $|x - a| < \varepsilon$

③ We proved that $\sup(a + B) = a + \sup(B) \quad \forall a \in A$

~~thus~~ Since $a + B \subset A + B$, then $\sup(A+B) \ge \sup(a+B) = a + \sup B \quad \forall a \in A$

Hence $\sup(A+B)$ is an upper bd of $A + \sup B$

Then $\sup(A+B) \ge \sup(A + \sup B) = \sup(A) + \sup B$

Conversely, it is clear that $a + b \le \sup(A) + \sup(B) \quad \forall a \in A,\ b \in B$

Since $\sup(A) + \sup(B)$ is an upper bd of $A+B$, then

$\sup(A+B) \le \sup(A) + \sup(B)$

Conclusion: $\sup(A) + \sup(B) = \sup(A+B)$

~~④ (a) By calculus $f(x) = x + x^{-1}$, $x > 0$ has min at $x = e^{-1}$, ... $\lim_{x\to\infty} f(x) = \infty$~~

~~Hence $\inf(A) = \min(A) = e + e^{-1}$, $\sup(A) = \infty$ ... $\inf(A)$ is NOT min~~

~~(b)~~

(4) (a) $f(x) = x + \frac{1}{x}$, $\cancel{x} > 0$

By calculus, $\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to \infty} f(x) = \infty$, $\min f(x) = 2$, at $x = 1$

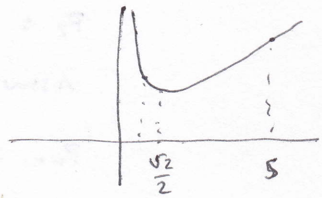
NOTE $1 \in \mathbb{Q}$

Hence $\inf(A) = \min(A) = 2$

A has NO s-p. $\sup(A) = \infty$

(b) $f(x) = x + \frac{1}{2x}$, $x > 0$. $\lim_{x \to 0^+} f(x) = \lim_{x \to \infty} f(x) = \infty$.

$\min f(x)$ occurs at $x = \frac{\sqrt{2}}{2} \notin \mathbb{Q}$



$\inf(B) = \frac{\sqrt{2}}{2} + (\sqrt{2})^{-1}$

B has NO MIN SINCE THE VALUE ABOVE IS NOT IN $\mathbb{Q}$

$\sup(B) = \max(B) = 5 + \frac{1}{10} = 5.1$ sup=MAX since value is in $\mathbb{Q}$

(c) $f(x) = xe^{-x}$, $\lim_{x \to -\infty} f(x) = -\infty$

$f'(x) = e^{-x}(1-x)$ $f(x)$ has local max at $x = 1$

Hence $\sup C = f(1) = e = \max(C)$

$\inf(C) = -\infty$. NO MIN

(5) $S = \{a_n = 1 - \frac{(-1)^n}{n}, n \in \mathbb{N}\}$ DIVIDE set into n even and n odd

$S_e = \{a_{2m} = 1 - \frac{1}{2m}, m \in \mathbb{N}\}$, $S_o = \{a_{2m-1} = 1 + \frac{1}{2m-1}, m \in \mathbb{N}\}$

$S = S_e \cup S_o$. $1 \geq |a_{2m}| \geq \frac{1}{2}$, $\forall m \in \mathbb{N}$, $a_2 = \frac{1}{2}$

$1 \leq |a_{2m-1}| \leq 2$ $\forall m \in \mathbb{N}$, $a_1 = 2$

Hence $\sup(S) = \max(S) = 2$

$\inf(S) = \min(S) = \frac{1}{2}$