

Since  $\triangle OAB \subset \text{sector } OAB \subset \triangle OAC$ , we have the same relationship for their areas:

$$\frac{\sin \theta}{2} < \frac{\theta}{2} < \frac{\tan \theta}{2} = \frac{\sin \theta}{2 \cos \theta}.$$

A manipulation of these inequalities yields

$$\cos \theta < \frac{\sin \theta}{\theta} < 1.$$

In particular,  $\cos \frac{1}{n} < n \sin \frac{1}{n} < 1$ . Moreover,

$$\cos\left(\frac{1}{n}\right) = \sqrt{1 - \sin^2\left(\frac{1}{n}\right)} > \sqrt{1 - \left(\frac{1}{n}\right)^2} > 1 - \frac{1}{n^2}.$$

However,

$$\lim_{n \rightarrow \infty} 1 - \frac{1}{n^2} = 1 = \lim_{n \rightarrow \infty} 1.$$

Therefore, by the Squeeze Theorem,  $\lim_{n \rightarrow \infty} n \sin \frac{1}{n} = 1$ .

### Exercises for Section 2.4

- A. In each of the following, compute the limit. Then, using  $\varepsilon = 10^{-6}$ , find an integer  $N$  that satisfies the limit definition.
- (a)  $\lim_{n \rightarrow \infty} \frac{\sin n^2}{\sqrt{n}}$     (b)  $\lim_{n \rightarrow \infty} \frac{1}{\log \log n}$     (c)  $\lim_{n \rightarrow \infty} \frac{3^n}{n!}$     (d)  $\lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{2n^2 - n + 2}$     (e)  $\lim_{n \rightarrow \infty} \sqrt{n^2 + n} - n$
- B. Show that  $\lim_{n \rightarrow \infty} \sin \frac{n\pi}{2}$  does not exist using the definition of limit.
- C. Prove that if  $a_n \leq b_n$  for  $n \geq 1$ ,  $L = \lim_{n \rightarrow \infty} a_n$ , and  $M = \lim_{n \rightarrow \infty} b_n$ , then  $L \leq M$ .
- D. Prove that if  $L = \lim_{n \rightarrow \infty} a_n$ , then  $L = \lim_{n \rightarrow \infty} a_{2n}$  and  $L = \lim_{n \rightarrow \infty} a_{n^2}$ .
- E. Sometimes, a limit is defined informally as follows: "As  $n$  goes to infinity,  $a_n$  gets closer and closer to  $L$ ." Find as many faults with this definition as you can.
- (a) Can a sequence satisfy this definition and still fail to converge?  
 (b) Can a sequence converge yet fail to satisfy this definition?
- F. Define a sequence  $(a_n)_{n=1}^{\infty}$  such that  $\lim_{n \rightarrow \infty} a_{n^2}$  exists but  $\lim_{n \rightarrow \infty} a_n$  does not exist.
- G. Suppose that  $\lim_{n \rightarrow \infty} a_n = L$  and  $L \neq 0$ . Prove there is some  $N$  such that  $a_n \neq 0$  for all  $n \geq N$ .
- H. Give a careful proof, using the definition of limit, that  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = M$  imply that  $\lim_{n \rightarrow \infty} 2a_n + 3b_n = 2L + 3M$ .
- I. For each  $x \in \mathbb{R}$ , determine whether  $\left(\frac{1}{1+x^n}\right)_{n=1}^{\infty}$  has a limit, and compute it when it exists.
- J. Let  $a_0$  and  $a_1$  be positive real numbers, and set  $a_{n+2} = \sqrt{a_{n+1}} + \sqrt{a_n}$  for  $n \geq 0$ .
- (a) Show that there is  $N$  such that for all  $n \geq N$ ,  $a_n \geq 1$ .  
 (b) Let  $\varepsilon_n = |a_n - 4|$ . Show that  $\varepsilon_{n+2} \leq (\varepsilon_{n+1} + \varepsilon_n)/3$  for  $n \geq N$ .  
 (c) Prove that this sequence converges.
- K. Show that the sequence  $(\log n)_{n=1}^{\infty}$  does not converge.

## 2.5 Basic Properties

## 2.5.1. PROPOSITION

then the set  $\{a_n : n \in \mathbb{N}\}$  is

PROOF. Let  $L = \lim_{n \rightarrow \infty} a_n$ . If

$N > 0$  such that  $|a_n - L| <$

$L -$

Let  $M = \max\{a_1, a_2, \dots\}$ .

Clearly, for all  $n$ , we have

It is also crucial that straightforward. The deta

## 2.5.2. THEOREM.

(i)  $\lim_{n \rightarrow \infty} a_n + b_n = L + M$

(ii)  $\lim_{n \rightarrow \infty} \alpha a_n = \alpha L$ ,

(iii)  $\lim_{n \rightarrow \infty} a_n b_n = LM$ , and

(iv)  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{M}$  if  $M \neq 0$ .

In the sequence  $(a_n)$

doing this because  $M \neq$

the 2.4(G). (We use "f

some  $N$  so that this hold

## Exercises for Secti

1. Prove Theorem 2.5.2.

2. Compute the following

(a)  $\lim_{n \rightarrow \infty} \frac{\sin n}{n}$

(b)  $\lim_{n \rightarrow \infty} \frac{\cos n}{n}$

(c) If  $\lim_{n \rightarrow \infty} a_n = L > 0$ , pro

Hint: Expre

(d) Let  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$

that  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = M$

(e) Find  $\lim_{n \rightarrow \infty} \frac{\log(2 + 3^n)}{2n}$

(f)  $\lim_{n \rightarrow \infty} (a_n)^{1/n} = \sqrt[n]{a_n} = 1$

Hint: Use the B

## 2.5 Basic Properties of Limits

**2.5.1. PROPOSITION.** If  $(a_n)_{n=1}^{\infty}$  is a convergent sequence of real numbers, then the set  $\{a_n : n \in \mathbb{N}\}$  is bounded.

**PROOF.** Let  $L = \lim_{n \rightarrow \infty} a_n$ . If we set  $\varepsilon = 1$ , then by the definition of limit, there is some  $N > 0$  such that  $|a_n - L| < 1$  for all  $n \geq N$ . In other words,

$$L - 1 < a_n < L + 1 \quad \text{for all } n \geq N.$$

Let  $M = \max\{a_1, a_2, \dots, a_{N-1}, L + 1\}$  and  $m = \min\{a_1, a_2, \dots, a_{N-1}, L - 1\}$ . Clearly, for all  $n$ , we have  $m \leq a_n \leq M$ . ■

It is also crucial that limits respect the arithmetic operations. Proving this is straightforward. The details are left as exercises.

**2.5.2. THEOREM.** If  $\lim_{n \rightarrow \infty} a_n = L$ ,  $\lim_{n \rightarrow \infty} b_n = M$ , and  $\alpha \in \mathbb{R}$ , then

- (1)  $\lim_{n \rightarrow \infty} (a_n + b_n) = L + M$ ,
- (2)  $\lim_{n \rightarrow \infty} \alpha a_n = \alpha L$ ,
- (3)  $\lim_{n \rightarrow \infty} a_n b_n = LM$ , and
- (4)  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{M}$  if  $M \neq 0$ .

In the sequence  $(a_n/b_n)_{n=1}^{\infty}$ , we ignore terms with  $b_n = 0$ . There is no problem doing this because  $M \neq 0$  implies that  $b_n \neq 0$  for all sufficiently large  $n$  (see Exercise 2.4.G). (We use “for all sufficiently large  $n$ ” as shorthand for saying there is some  $N$  so that this holds for all  $n \geq N$ .)

## Exercises for Section 2.5

- A. Prove Theorem 2.5.2. HINT: For part (4), first bound the denominator away from 0.
- B. Compute the following limits.
  - (a)  $\lim_{n \rightarrow \infty} \frac{\tan \frac{\pi}{n}}{n \sin^2 \frac{\pi}{2n}}$
  - (b)  $\lim_{n \rightarrow \infty} \frac{2^{100+5n}}{e^{4n-10}}$
  - (c)  $\lim_{n \rightarrow \infty} \frac{\csc \frac{1}{n} + 2 \arctan n}{\log n}$
- C. If  $\lim_{n \rightarrow \infty} a_n = L > 0$ , prove that  $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{L}$ . Be sure to discuss the issue of when  $\sqrt{a_n}$  makes sense. HINT: Express  $|\sqrt{a_n} - \sqrt{L}|$  in terms of  $|a_n - L|$ .
- D. Let  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  be two sequences of real numbers such that  $|a_n - b_n| < \frac{1}{n}$ . Suppose that  $L = \lim_{n \rightarrow \infty} a_n$  exists. Show that  $(b_n)_{n=1}^{\infty}$  converges to  $L$  also.
- E. Find  $\lim_{n \rightarrow \infty} \frac{\log(2+3^n)}{2n}$ . HINT:  $\log(2+3^n) = \log 3^n + \log \frac{2+3^n}{3^n}$
- F. (a) Let  $x_n = \sqrt[n]{n} - 1$ . Use the fact that  $(1+x_n)^n = n$  to show that  $x_n^2 \leq 2/n$ . HINT: Use the Binomial Theorem and throw away most terms.

The following easy corollary of the Monotone Convergence Theorem is again a reflection of the completeness of the real numbers. This is just the tool needed to establish the key result of the next section, the Bolzano–Weierstrass Theorem (2.7.2).

Again, the corresponding result for intervals of rational numbers is false. See Example 2.7.6. The result would also be false if we changed closed intervals to open intervals. For example,  $\bigcap_{n \geq 1} (0, \frac{1}{n}) = \emptyset$ .

### 2.6.3. NESTED INTERVALS LEMMA.

Suppose that  $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$  are nonempty closed intervals such that  $I_{n+1} \subseteq I_n$  for each  $n \geq 1$ . Then the intersection  $\bigcap_{n \geq 1} I_n$  is nonempty.

**PROOF.** Notice that since  $I_{n+1}$  is contained in  $I_n$ , it follows that

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n.$$

Thus  $(a_n)$  is a monotone increasing sequence bounded above by  $b_1$ ; and likewise  $(b_n)$  is a monotone decreasing sequence bounded below by  $a_1$ . Hence by Theorem 2.6.1,  $a = \lim_{n \rightarrow \infty} a_n$  exists, as does  $b = \lim_{n \rightarrow \infty} b_n$ . By Exercise 2.4.C,  $a \leq b$ . Thus

$$a_k \leq a \leq b \leq b_k.$$

Consequently, the point  $a$  belongs to  $I_k$  for each  $k \geq 1$ . ■

### Exercises for Section 2.6

- A. Say that  $\lim_{n \rightarrow \infty} a_n = +\infty$  if for every  $R \in \mathbb{R}$ , there is an integer  $N$  such that  $a_n > R$  for all  $n \geq N$ . Show that a divergent monotone increasing sequence converges to  $+\infty$  in this sense.
- B. Let  $a_1 = 0$  and  $a_{n+1} = \sqrt{5+2a_n}$  for  $n \geq 1$ . Show that  $\lim_{n \rightarrow \infty} a_n$  exists and find the limit.
- C. Is  $S = \{x \in \mathbb{R} : 0 < \sin(\frac{1}{x}) < \frac{1}{2}\}$  bounded above (below)? If so, find  $\sup S$  ( $\inf S$ ).
- D. Evaluate  $\lim_{n \rightarrow \infty} \sqrt[n]{3^n + 5^n}$ .
- E. Suppose  $(a_n)$  is a sequence of positive real numbers such that  $a_{n+1} - 2a_n + a_{n-1} > 0$  for all  $n \geq 1$ . Prove that the sequence either converges or tends to  $+\infty$ .
- F. Let  $a, b$  be positive real numbers. Set  $x_0 = a$  and  $x_{n+1} = (x_n^{-1} + b)^{-1}$  for  $n \geq 0$ .
- (a) Prove that  $x_n$  is monotone decreasing.  
(b) Prove that the limit exists and find it.
- G. Let  $a_n = (\sum_{k=1}^n 1/k) - \log n$  for  $n \geq 1$ . **Euler's constant** is defined as  $\gamma = \lim_{n \rightarrow \infty} a_n$ . Show that  $(a_n)_{n=1}^{\infty}$  is decreasing and bounded below by zero, and so this limit exists.  
HINT: Prove that  $1/(n+1) \leq \log(n+1) - \log n \leq 1/n$ .
- H. Let  $x_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}}$ .
- (a) Show that  $x_n < x_{n+1}$ .  
(b) Show that  $x_{n+1}^2 \leq 1 + \sqrt{2}x_n$ . HINT: Square  $x_{n+1}$  and factor a 2 out of the square root.  
(c) Hence show that  $x_n$  is bounded above by 2. Deduce that  $\lim_{n \rightarrow \infty} x_n$  exists.

- I. (a) Let  $(a_n)_{n=1}^{\infty}$  be a bounded sequence. Prove that  $(b_n)$  converges. This  
(b) Without redoing the proof, conclude that  $\liminf a_n := \lim_{n \rightarrow \infty} (\inf_{k \geq n} a_k)$   
(c) Extend the definitions of  $\liminf$  and  $\limsup$  to unbounded sequences. Give an example with  $\limsup a_n = +\infty$ .
- J. Show that  $(a_n)_{n=1}^{\infty}$  converges to  $+\infty$  if and only if  $(1/a_n)_{n=1}^{\infty}$  converges to 0.
- K. If a sequence  $(a_n)$  is not bounded above, should  $\limsup a_n$  be? Formulate a precise definition of  $\limsup a_n$ .
- L. Suppose  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are bounded sequences and  $\lim_{n \rightarrow \infty} (a_n + b_n) = L$  exists. Prove that  $\limsup a_n + \limsup b_n = L$ .
- M. Suppose that  $(a_n)_{n=1}^{\infty}$  has  $a_n > 0$  for all  $n$ . Prove that  $\limsup a_n = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}$ .
- N. Suppose  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are bounded sequences. Prove that there is a constant  $M$  such that  $|a_n b_n| \leq M$  for all  $n$ .

## 2.7 Subsequences

Given one sequence, we can build a new one, called a subsequence, by picking out some of the terms of the original sequence. If the original sequence does not converge, it is possible that a subsequence does.

### 2.7.1. DEFINITION. A subsequence

$(a_{n_k})_{k=1}^{\infty} = (a_{n_1}, a_{n_2}, a_{n_3}, \dots)$ , where  $n_1 < n_2 < n_3 < \dots$  and  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ .

For example,  $(a_{2k})_{k=1}^{\infty}$  and  $(a_{k^2})_{k=1}^{\infty}$  are subsequences of  $(a_n)_{n=1}^{\infty}$ . Notice that if  $(a_n)_{n=1}^{\infty}$  converges to  $L$ , then every subsequence converges to the same limit  $L$ .

It is easy to verify that if  $(a_n)_{n=1}^{\infty}$  converges to the same limit  $L$ , then every subsequence converges to the same limit  $L$ . However, we will show that if a sequence does not converge, then there is a subsequence that does converge.

### 2.7.2. BOLZANO–WEIERSTRASS THEOREM

Every bounded sequence of real numbers has a convergent subsequence.

**PROOF.** Let  $(a_n)$  be a bounded sequence. Consider the whole (infinite) sequence. No matter what, it has a limit point  $L$ . If  $L$  is a limit point of the sequence  $(a_n)$ , then  $L$  is a limit point of the sequence  $(a_n)$ , and  $L$  is a limit point of the sequence  $(a_n)$ .

So let  $I_1 = [-B, B]$ . Split it into two halves,  $[0, B]$  and  $[-B, 0]$ . One of these halves contains a limit point. Similarly, divide  $I_2$  into two halves.

Iterating this, we obtain  $b_2^2 - 8 < 32^{-1}$ ,  $b_3^2 - 8 < 32^{-3}$ , and  $b_4^2 - 8 < 32^{-7}$ . In general, we establish by induction that

$$0 < b_n^2 - 8 < 32^{1-2^{n-1}}.$$

Since  $b_n$  is positive and  $b^2 - 8 = (b - \sqrt{8})(b + \sqrt{8})$ , it follows that

$$0 < b_n - \sqrt{8} = \frac{b_n^2 - 8}{b_n + \sqrt{8}} < \frac{32^{1-2^{n-1}}}{2\sqrt{8}} < 6(32^{-2^{n-1}}).$$

Lastly, using the fact that  $32^2 = 1024 > 10^3$ , we obtain

$$0 < b_n - \sqrt{8} < 10 \cdot 10^{-3 \cdot 2^{n-2}}.$$

In particular,  $\lim_{n \rightarrow \infty} b_n = \sqrt{8}$ . In fact, the convergence is so rapid that  $b_{10}$  approximates  $\sqrt{8}$  to more than 750 digits of accuracy. See Example 11.2.2 for a more general analysis in terms of Newton's method.

Let  $a_n = 8/b_n$ . Then  $a_n$  is monotone increasing to  $\sqrt{8}$ . Both  $a_n$  and  $b_n$  are rational, but  $\sqrt{8}$  is irrational. Thus the sets  $J_n = \{x \in \mathbb{Q} : a_n \leq x \leq b_n\}$  form a decreasing sequence of nonempty intervals of rational numbers with empty intersection.

## Exercises for Section 2.7

- A. Show that  $(a_n) = \left(\frac{n \cos^n(n)}{\sqrt{n^2+2n}}\right)_{n=1}^{\infty}$  has a convergent subsequence.
- B. Does the sequence  $(b_n) = (n + \cos(n\pi)\sqrt{n^2+1})_{n=1}^{\infty}$  have a convergent subsequence?
- C. Does the sequence  $(a_n) = (\cos \log n)_{n=1}^{\infty}$  converge?
- D. Show that every sequence has a monotone subsequence.
- E. Use trig identities to show that  $|\sin x - \sin y| \leq |x - y|$ .  
HINT: Let  $a = (x+y)/2$  and  $b = (x-y)/2$ . Use the addition formula for  $\sin(a \pm b)$ .
- F. Define  $x_1 = 2$  and  $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$  for  $n \geq 1$ .  
(a) Find a formula for  $x_{n+1}^2 - 5$  in terms of  $x_n^2 - 5$ .  
(b) Hence evaluate  $\lim_{n \rightarrow \infty} x_n$ .  
(c) Compute the first ten terms on a computer or a calculator.  
(d) Show that the tenth term approximates the limit to over 600 decimal places.
- G. Let  $(x_n)_{n=1}^{\infty}$  be a sequence of real numbers. Suppose that there is a real number  $L$  such that  $L = \lim_{n \rightarrow \infty} x_{3n-1} = \lim_{n \rightarrow \infty} x_{3n+1} = \lim_{n \rightarrow \infty} x_{3n}$ . Show that  $\lim_{n \rightarrow \infty} x_n$  exists and equals  $L$ .
- H. Let  $(x_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{R}$ . Suppose there is a number  $L$  such that every subsequence  $(x_{n_k})_{k=1}^{\infty}$  has a subsubsequence  $(x_{n_{k(l)}})_{l=1}^{\infty}$  with  $\lim_{l \rightarrow \infty} x_{n_{k(l)}} = L$ . Show that the whole sequence converges to  $L$ . HINT: If not, you could find a subsequence bounded away from  $L$ .
- I. Suppose  $(x_n)_{n=1}^{\infty}$  is a sequence in  $\mathbb{R}$ , and that  $L_k$  are real numbers with  $\lim_{k \rightarrow \infty} L_k = L$ . If for each  $k \geq 1$ , there is a subsequence of  $(x_n)_{n=1}^{\infty}$  converging to  $L_k$ , show that some subsequence converges to  $L$ . HINT: Find an increasing sequence  $n_k$  such that  $|x_{n_k} - L| < 1/k$ .

J. (a) Suppose that  $(x_n)_{n=1}^{\infty}$  is a subsequence  $(x_{n_k})_{k=1}^{\infty}$  such that  $\lim_{k \rightarrow \infty} x_{n_k} = L$ .  
(b) Similarly, prove that there is a subsequence  $(x_{n_k})_{k=1}^{\infty}$  such that  $\lim_{k \rightarrow \infty} x_{n_k} = L$ .

K. Let  $(x_n)_{n=1}^{\infty}$  be an arbitrary sequence. Show that either  $\lim_{n \rightarrow \infty} x_n = \infty$  or  $\lim_{n \rightarrow \infty} x_n = -\infty$  or  $\lim_{n \rightarrow \infty} x_n = L$  for some real number  $L$ .

L. Construct a sequence  $(x_n)_{n=1}^{\infty}$  with  $\lim_{n \rightarrow \infty} x_n = L$ .

## 2.8 Cauchy Sequences

Can we decide whether a sequence has a limit? To do this, we need a convergence test that does not require knowing the limit. This section shows which sequences are a subset of  $\mathbb{R}$  being complete. We will see that the real numbers converge actually do. As we will see, the real numbers by our construction are complete. To obtain an appropriate definition, we consider the terms get close to each other.

### 2.8.1. PROPOSITION.

If  $(x_n)_{n=1}^{\infty}$  is a Cauchy sequence, then there is an integer  $N$  such that for all  $m, n > N$ ,  $|x_m - x_n| < \epsilon$ .

PROOF. Fix  $\epsilon > 0$  and use the definition of a Cauchy sequence to find an integer  $N$  such that  $|x_m - x_n| < \epsilon$  for all  $m, n > N$ .

It is clear that  $|x_n - x_n| = 0 < \epsilon$  for all  $n > N$ .

In order for  $N$  to work in the definition, it is not enough to just find an  $N$  for a fixed  $\epsilon$ .

We make the conclusion that the definition of a Cauchy sequence is that for every  $\epsilon > 0$ , there is an integer  $N$  such that for all  $m, n > N$ ,  $|x_m - x_n| < \epsilon$ .

### 2.8.2. DEFINITION.

A sequence  $(x_n)_{n=1}^{\infty}$  is said to be a Cauchy sequence if for every  $\epsilon > 0$ , there is an integer  $N$  such that for all  $m, n > N$ ,  $|x_m - x_n| < \epsilon$ .

- J. (a) Suppose that  $(x_n)_{n=1}^{\infty}$  is a sequence of real numbers. If  $L = \liminf x_n$ , show that there is a subsequence  $(x_{n_k})_{k=1}^{\infty}$  such that  $\lim_{k \rightarrow \infty} x_{n_k} = L$ .
- (b) Similarly, prove that there is a subsequence  $(x_{n_l})_{l=1}^{\infty}$  such that  $\lim_{l \rightarrow \infty} x_{n_l} = \limsup x_n$ .
- K. Let  $(x_n)_{n=1}^{\infty}$  be an arbitrary sequence. Prove that there is a subsequence  $(x_{n_k})_{k=1}^{\infty}$  which converges or  $\lim_{k \rightarrow \infty} x_{n_k} = \infty$  or  $\lim_{k \rightarrow \infty} x_{n_k} = -\infty$ .
- L. Construct a sequence  $(x_n)_{n=1}^{\infty}$  such that for every real number  $L$ , there is a subsequence  $(x_{n_k})_{k=1}^{\infty}$  with  $\lim_{k \rightarrow \infty} x_{n_k} = L$ .

## 2.8 Cauchy Sequences

Can we decide whether a sequence converges *without* first finding the value of the limit? To do this, we need an intrinsic property of a sequence which is equivalent to convergence that does not make use of the value of the limit. This intrinsic property shows which sequences are 'supposed' to converge. This leads us to the notion of a subset of  $\mathbb{R}$  being *complete* if all sequences in the subset that are 'supposed' to converge actually do. As we shall see, this completeness property has been built into the real numbers by our construction of infinite decimals.

To obtain an appropriate condition, notice that if a sequence  $(a_n)$  converges to  $L$ , then as the terms get close to the limit, they are getting close to each other.

**2.8.1. PROPOSITION.** *Let  $(a_n)_{n=1}^{\infty}$  be a sequence converging to  $L$ . For every  $\varepsilon > 0$ , there is an integer  $N$  such that*

$$|a_n - a_m| < \varepsilon \quad \text{for all } m, n \geq N.$$

**PROOF.** Fix  $\varepsilon > 0$  and use the value  $\varepsilon/2$  in the definition of limit. Then there is an

implies the Least Upper Bound Principle, go through our proof to obtain an increasing sequence of lower bounds,  $y_k$ , and a decreasing sequence of elements  $x_k \in S$  with  $x_k < y_k + 10^{-k}$ . Show that the sequence  $x_1, y_1, x_2, y_2, \dots$  is Cauchy. The limit  $L$  will be the greatest lower bound. Fill in the details yourself (Exercise 2.8.G).

### Exercises for Section 2.8

- A. Let  $(x_n)$  be Cauchy with a subsequence  $(x_{n_k})$  such that  $\lim_{k \rightarrow \infty} x_{n_k} = a$ . Show that  $\lim_{n \rightarrow \infty} x_n = a$ .
- B. Give a sequence  $(a_n)$  such that  $\lim_{n \rightarrow \infty} |a_n - a_{n+1}| = 0$ , but the sequence does not converge.
- C. Let  $(a_n)$  be a sequence such that  $\lim_{N \rightarrow \infty} \sum_{n=1}^N |a_n - a_{n+1}| < \infty$ . Show that  $(a_n)$  is Cauchy.
- D. If  $(x_n)_{n=1}^{\infty}$  is Cauchy, show that it has a subsequence  $(x_{n_k})$  such that  $\sum_{k=1}^{\infty} |x_{n_k} - x_{n_{k+1}}| < \infty$ .
- E. Suppose that  $(a_n)$  is a sequence such that  $a_{2n} \leq a_{2n+2} \leq a_{2n+3} \leq a_{2n+1}$  for all  $n \geq 0$ . Show that this sequence is Cauchy if and only if  $\lim_{n \rightarrow \infty} |a_n - a_{n+1}| = 0$ .
- F. Give an example of a sequence  $(a_n)$  such that  $a_{2n} \leq a_{2n+2} \leq a_{2n+3} \leq a_{2n+1}$  for all  $n \geq 0$  which does not converge.
- G. Fill in the details of how the Completeness Theorem implies the Least Upper Bound Principle.
- H. Let  $a_0 = 0$  and set  $a_{n+1} = \cos(a_n)$  for  $n \geq 0$ . Try this on your calculator (use radian mode!).
- (a) Show that  $a_{2n} \leq a_{2n+2} \leq a_{2n+3} \leq a_{2n+1}$  for all  $n \geq 0$ .
- (b) Use the Mean Value Theorem to find an explicit number  $r < 1$  such that  $|a_{n+2} - a_{n+1}| \leq r|a_n - a_{n+1}|$  for all  $n \geq 0$ . Hence show that this sequence is Cauchy.
- (c) Describe the limit geometrically as the intersection point of two curves.

- I. Evaluate the continued fraction

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

- J. Let  $x_0 = 0$  and  $x_{n+1} = \sqrt{5 - 2x_n}$  for  $n \geq 0$ . Show that this sequence converges and compute the limit. HINT: Show that the even terms increase and the odd terms decrease.
- K. Consider an infinite binary expansion  $(0.e_1e_2e_3\dots)_{\text{base } 2}$ , where each  $e_i \in \{0, 1\}$ . Show that  $a_n = \sum_{i=1}^n 2^{-i}e_i$  is Cauchy for every choice of zeros and ones.
- L. One base-independent construction of the real numbers uses Cauchy sequences of rational numbers. This exercise asks for the definitions that go into such a proof.
- (a) Find a way to decide when two Cauchy sequences should determine the same real number without using their limits. HINT: Combine the two sequences into one.
- (b) Your definition in (a) should be an equivalence relation. Is it? (See Appendix 1.3.)
- (c) How are addition and multiplication defined?
- (d) How is the order defined?

### 2.9 Countable Sets

Cardinality measures the size of a set in the crudest of ways—by counting the numbers of elements. Obviously, the number of elements in a set could be 0, 1, 2, 3, 4, or some other finite number. Or a set can have infinitely many elements. Perhaps