Since $\triangle OAB \subset$ sector $OAB \subset \triangle OAC$, we have the same relationship for their areas:

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2} = \frac{\sin\theta}{2\cos\theta}.$$

A manipulation of these inequalities yields

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

In particular, $\cos\frac{1}{n} < n\sin\frac{1}{n} < 1$. Moreover,

$$\cos\left(\tfrac{1}{n}\right) = \sqrt{1-\sin^2\left(\tfrac{1}{n}\right)} > \sqrt{1-\left(\tfrac{1}{n}\right)^2} > 1-\frac{1}{n^2}.$$

However,

$$\lim_{n\to\infty} 1 - \frac{1}{n^2} = 1 = \lim_{n\to\infty} 1.$$

Therefore, by the Squeeze Theorem, $\lim\limits_{n\to\infty} n\sin\frac{1}{n} = 1$.

Exercises for Section 2.4

A. In each of the following, compute the limit. Then, using $\varepsilon = 10^{-6}$, find an integer N that satisfies the limit definition.
(a) $\lim\limits_{n\to\infty} \dfrac{\sin n^2}{\sqrt{n}}$ (b) $\lim\limits_{n\to\infty} \dfrac{1}{\log\log n}$ (c) $\lim\limits_{n\to\infty} \dfrac{3^n}{n!}$ (d) $\lim\limits_{n\to\infty} \dfrac{n^2+2n+1}{2n^2-n+2}$ (e) $\lim\limits_{n\to\infty} \sqrt{n^2+n} -$
n

B. Show that $\lim\limits_{n\to\infty} \sin\frac{n\pi}{2}$ does not exist using the definition of limit.

C. Prove that if $a_n \le b_n$ for $n \ge 1$, $L = \lim\limits_{n\to\infty} a_n$, and $M = \lim\limits_{n\to\infty} b_n$, then $L \le M$.

D. Prove that if $L = \lim\limits_{n\to\infty} a_n$, then $L = \lim\limits_{n\to\infty} a_{2n}$ and $L = \lim\limits_{n\to\infty} a_{n^2}$.

E. Sometimes, a limit is defined informally as follows: "As n goes to infinity, a_n gets closer and closer to L." Find as many faults with this definition as you can.
(a) Can a sequence satisfy this definition and still fail to converge?
(b) Can a sequence converge yet fail to satisfy this definition?

F. Define a sequence $(a_n)_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} a_{n^2}$ exists but $\lim\limits_{n\to\infty} a_n$ does not exist.

G. Suppose that $\lim\limits_{n\to\infty} a_n = L$ and $L \ne 0$. Prove there is some N such that $a_n \ne 0$ for all $n \ge N$.

H. Give a careful proof, using the definition of limit, that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$ imply that $\lim\limits_{n\to\infty} 2a_n + 3b_n = 2L + 3M$.

I. For each $x \in \mathbb{R}$, determine whether $\left(\dfrac{1}{1+x^n}\right)_{n=1}^{\infty}$ has a limit, and compute it when it exists.

J. Let a_0 and a_1 be positive real numbers, and set $a_{n+2} = \sqrt{a_{n+1}} + \sqrt{a_n}$ for $n \ge 0$.
(a) Show that there is N such that for all $n \ge N$, $a_n \ge 1$.
(b) Let $\varepsilon_n = |a_n - 4|$. Show that $\varepsilon_{n+2} \le (\varepsilon_{n+1} + \varepsilon_n)/3$ for $n \ge N$.
(c) Prove that this sequence converges.

K. Show that the sequence $(\log n)_{n=1}^{\infty}$ does not converge.

2.5 Basic Properties of Limits

2.5.1. PROPOSITION. *If $(a_n)_{n=1}^{\infty}$ is a convergent sequence of real numbers, then the set $\{a_n : n \in \mathbb{N}\}$ is bounded.*

PROOF. Let $L = \lim_{n\to\infty} a_n$. If we set $\varepsilon = 1$, then by the definition of limit, there is some $N > 0$ such that $|a_n - L| < 1$ for all $n \geq N$. In other words,

$$L - 1 < a_n < L + 1 \quad \text{for all} \quad n \geq N.$$

Let $M = \max\{a_1, a_2, \ldots, a_{N-1}, L+1\}$ and $m = \min\{a_1, a_2, \ldots, a_{N-1}, L-1\}$. Clearly, for all n, we have $m \leq a_n \leq M$. ■

It is also crucial that limits respect the arithmetic operations. Proving this is straightforward. The details are left as exercises.

2.5.2. THEOREM. *If $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$, and $\alpha \in \mathbb{R}$, then*

(1) $\lim_{n\to\infty} a_n + b_n = L + M$,

(2) $\lim_{n\to\infty} \alpha a_n = \alpha L$,

(3) $\lim_{n\to\infty} a_n b_n = LM$, *and*

(4) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{M}$ *if $M \neq 0$.*

In the sequence $(a_n/b_n)_{n=1}^{\infty}$, we ignore terms with $b_n = 0$. There is no problem doing this because $M \neq 0$ implies that $b_n \neq 0$ for all sufficiently large n (see Exercise 2.4.G). (We use "for all sufficiently large n" as shorthand for saying there is some N so that this holds for all $n \geq N$.)

Exercises for Section 2.5

A. Prove Theorem 2.5.2. HINT: For part (4), first bound the denominator away from 0.

B. Compute the following limits.

(a) $\lim_{n\to\infty} \dfrac{\tan\frac{\pi}{n}}{n\sin^2\frac{2}{n}}$ (b) $\lim_{n\to\infty} \dfrac{2^{100+5n}}{e^{4n-10}}$ (c) $\lim_{n\to\infty} \dfrac{\csc\frac{1}{n}}{n} + \dfrac{2\arctan n}{\log n}$

C. If $\lim_{n\to\infty} a_n = L > 0$, prove that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$. Be sure to discuss the issue of when $\sqrt{a_n}$ makes sense. HINT: Express $|\sqrt{a_n} - \sqrt{L}|$ in terms of $|a_n - L|$.

D. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of real numbers such that $|a_n - b_n| < \frac{1}{n}$. Suppose that $L = \lim_{n\to\infty} a_n$ exists. Show that $(b_n)_{n=1}^{\infty}$ converges to L also.

E. Find $\lim_{n\to\infty} \dfrac{\log(2+3^n)}{2n}$. HINT: $\log(2+3^n) = \log 3^n + \log\frac{2+3^n}{3^n}$

F. (a) Let $x_n = \sqrt[n]{n} - 1$. Use the fact that $(1+x_n)^n = n$ to show that $x_n^2 \leq 2/n$.
HINT: Use the Binomial Theorem and throw away most terms.

The following easy corollary of the Monotone Convergence Theorem is again a reflection of the completeness of the real numbers. This is just the tool needed to establish the key result of the next section, the Bolzano–Weierstrass Theorem (2.7.2).

Again, the corresponding result for intervals of rational numbers is false. See Example 2.7.6. The result would also be false if we changed closed intervals to open intervals. For example, $\bigcap_{n\geq 1}(0,\frac{1}{n}) = \varnothing$.

2.6.3. NESTED INTERVALS LEMMA.
Suppose that $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$ are nonempty closed intervals such that $I_{n+1} \subseteq I_n$ for each $n \geq 1$. Then the intersection $\bigcap_{n\geq 1} I_n$ is nonempty.

PROOF. Notice that since I_{n+1} is contained in I_n, it follows that

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n.$$

Thus (a_n) is a monotone increasing sequence bounded above by b_1; and likewise (b_n) is a monotone decreasing sequence bounded below by a_1. Hence by Theorem 2.6.1, $a = \lim\limits_{n\to\infty} a_n$ exists, as does $b = \lim\limits_{n\to\infty} b_n$. By Exercise 2.4.C, $a \leq b$. Thus

$$a_k \leq a \leq b \leq b_k.$$

Consequently, the point a belongs to I_k for each $k \geq 1$. ■

Exercises for Section 2.6

A. Say that $\lim\limits_{n\to\infty} a_n = +\infty$ if for every $R \in \mathbb{R}$, there is an integer N such that $a_n > R$ for all $n \geq N$. Show that a divergent monotone increasing sequence converges to $+\infty$ in this sense.

B. Let $a_1 = 0$ and $a_{n+1} = \sqrt{5 + 2a_n}$ for $n \geq 1$. Show that $\lim\limits_{n\to\infty} a_n$ exists and find the limit.

C. Is $S = \{x \in \mathbb{R} : 0 < \sin(\frac{1}{x}) < \frac{1}{2}\}$ bounded above (below)? If so, find $\sup S$ ($\inf S$).

D. Evaluate $\lim\limits_{n\to\infty} \sqrt[n]{3^n + 5^n}$.

E. Suppose (a_n) is a sequence of positive real numbers such that $a_{n+1} - 2a_n + a_{n-1} > 0$ for all $n \geq 1$. Prove that the sequence either converges or tends to $+\infty$.

F. Let a, b be positive real numbers. Set $x_0 = a$ and $x_{n+1} = (x_n^{-1} + b)^{-1}$ for $n \geq 0$.

(a) Prove that x_n is monotone decreasing.
(b) Prove that the limit exists and find it.

G. Let $a_n = (\sum_{k=1}^{n} 1/k) - \log n$ for $n \geq 1$. **Euler's constant** is defined as $\gamma = \lim\limits_{n\to\infty} a_n$. Show that $(a_n)_{n=1}^{\infty}$ is decreasing and bounded below by zero, and so this limit exists.
HINT: Prove that $1/(n+1) \leq \log(n+1) - \log n \leq 1/n$.

H. Let $x_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}}$.

(a) Show that $x_n < x_{n+1}$.
(b) Show that $x_{n+1}^2 \leq 1 + \sqrt{2}x_n$. HINT: Square x_{n+1} and factor a 2 out of the square root.
(c) Hence show that x_n is bounded above by 2. Deduce that $\lim\limits_{n\to\infty} x_n$ exists.

Iterating this, we obtain $b_2^2 - 8 < 32^{-1}$, $b_3^2 - 8 < 32^{-3}$, and $b_4^2 - 8 < 32^{-7}$. In general, we establish by induction that

$$0 < b_n^2 - 8 < 32^{1-2^{n-1}}.$$

Since b_n is positive and $b^2 - 8 = (b - \sqrt{8})(b + \sqrt{8})$, it follows that

$$0 < b_n - \sqrt{8} = \frac{b_n^2 - 8}{b_n + \sqrt{8}} < \frac{32^{1-2^{n-1}}}{2\sqrt{8}} < 6(32^{-2^{n-1}}).$$

Lastly, using the fact that $32^2 = 1024 > 10^3$, we obtain

$$0 < b_n - \sqrt{8} < 10 \cdot 10^{-3 \cdot 2^{n-2}}.$$

In particular, $\lim_{n \to \infty} b_n = \sqrt{8}$. In fact, the convergence is so rapid that b_{10} approximates $\sqrt{8}$ to more than 750 digits of accuracy. See Example 11.2.2 for a more general analysis in terms of Newton's method.

Let $a_n = 8/b_n$. Then a_n is monotone increasing to $\sqrt{8}$. Both a_n and b_n are rational, but $\sqrt{8}$ is irrational. Thus the sets $J_n = \{x \in \mathbb{Q} : a_n \le x \le b_n\}$ form a decreasing sequence of nonempty intervals of *rational* numbers with empty intersection.

Exercises for Section 2.7

A. Show that $(a_n) = \left(\frac{n \cos^n(n)}{\sqrt{n^2+2n}}\right)_{n=1}^{\infty}$ has a convergent subsequence.

B. Does the sequence $(b_n) = \left(n + \cos(n\pi)\sqrt{n^2+1}\right)_{n=1}^{\infty}$ have a convergent subsequence?

C. Does the sequence $(a_n) = (\cos \log n)_{n=1}^{\infty}$ converge?

D. Show that every sequence has a monotone subsequence.

E. Use trig identities to show that $|\sin x - \sin y| \le |x - y|$.
HINT: Let $a = (x+y)/2$ and $b = (x-y)/2$. Use the addition formula for $\sin(a \pm b)$.

F. Define $x_1 = 2$ and $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$ for $n \ge 1$.

(a) Find a formula for $x_{n+1}^2 - 5$ in terms of $x_n^2 - 5$.
(b) Hence evaluate $\lim_{n \to \infty} x_n$.
(c) Compute the first ten terms on a computer or a calculator.
(d) Show that the tenth term approximates the limit to over 600 decimal places.

G. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that there is a real number L such that $L = \lim_{n \to \infty} x_{3n-1} = \lim_{n \to \infty} x_{3n+1} = \lim_{n \to \infty} x_{3n}$. Show that $\lim_{n \to \infty} x_n$ exists and equals L.

H. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Suppose there is a number L such that every subsequence $(x_{n_k})_{k=1}^{\infty}$ has a subsubsequence $(x_{n_{k(l)}})_{l=1}^{\infty}$ with $\lim_{l \to \infty} x_{n_{k(l)}} = L$. Show that the whole sequence converges to L. HINT: If not, you could find a subsequence bounded away from L.

I. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$, and that L_k are real numbers with $\lim_{k \to \infty} L_k = L$. If for each $k \ge 1$, there is a subsequence of $(x_n)_{n=1}^{\infty}$ converging to L_k, show that some subsequence converges to L. HINT: Find an increasing sequence n_k such that $|x_{n_k} - L| < 1/k$.

J. (a) Suppose that $(x_n)_{n=1}^\infty$ is a sequence of real numbers. If $L = \liminf x_n$, show that there is a subsequence $(x_{n_k})_{k=1}^\infty$ such that $\lim_{k\to\infty} x_{n_k} = L$.

(b) Similarly, prove that there is a subsequence $(x_{n_l})_{l=1}^\infty$ such that $\lim_{l\to\infty} x_{n_l} = \limsup x_n$.

K. Let $(x_n)_{n=1}^\infty$ be an arbitrary sequence. Prove that there is a subsequence $(x_{n_k})_{k=1}^\infty$ which converges or $\lim_{k\to\infty} x_{n_k} = \infty$ or $\lim_{k\to\infty} x_{n_k} = -\infty$.

L. Construct a sequence $(x_n)_{n=1}^\infty$ such that for every real number L, there is a subsequence $(x_{n_k})_{k=1}^\infty$ with $\lim_{k\to\infty} x_{n_k} = L$.

2.8 Cauchy Sequences

Can we decide whether a sequence converges *without* first finding the value of the limit? To do this, we need an intrinsic property of a sequence which is equivalent to convergence that does not make use of the value of the limit. This intrinsic property shows which sequences are ‘supposed’ to converge. This leads us to the notion of a subset of $\mathbb{R}$ being *complete* if all sequences in the subset that are ‘supposed’ to converge actually do. As we shall see, this completeness property has been built into the real numbers by our construction of infinite decimals.

To obtain an appropriate condition, notice that if a sequence (a_n) converges to L, then as the terms get close to the limit, they are getting close to each other.

2.8.1. PROPOSITION. *Let $(a_n)_{n=1}^\infty$ be a sequence converging to L. For every $\varepsilon > 0$, there is an integer N such that*

$$|a_n - a_m| < \varepsilon \quad \textit{for all} \quad m,n \geq N.$$

PROOF. Fix $\varepsilon > 0$ and use the value $\varepsilon/2$ in the definition of limit. Then there is an implies the Least Upper Bound Principle, go through our proof to obtain an increasing sequence of lower bounds, y_k, and a decreasing sequence of elements $x_k \in S$ with $x_k < y_k + 10^{-k}$. Show that the sequence $x_1, y_1, x_2, y_2, \ldots$ is Cauchy. The limit L will be the greatest lower bound. Fill in the details yourself (Exercise 2.8.G).

Exercises for Section 2.8

A. Let (x_n) be Cauchy with a subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = a$. Show that $\lim_{n\to\infty} x_n = a$.

B. Give a sequence (a_n) such that $\lim_{n\to\infty} |a_n - a_{n+1}| = 0$, but the sequence does not converge.

C. Let (a_n) be a sequence such that $\lim_{N\to\infty} \sum_{n=1}^{N} |a_n - a_{n+1}| < \infty$. Show that (a_n) is Cauchy.

D. If $(x_n)_{n=1}^{\infty}$ is Cauchy, show that it has a subsequence (x_{n_k}) such that $\sum_{k=1}^{\infty} |x_{n_k} - x_{n_{k+1}}| < \infty$.

E. Suppose that (a_n) is a sequence such that $a_{2n} \le a_{2n+2} \le a_{2n+3} \le a_{2n+1}$ for all $n \ge 0$. Show that this sequence is Cauchy if and only if $\lim_{n\to\infty} |a_n - a_{n+1}| = 0$.

F. Give an example of a sequence (a_n) such that $a_{2n} \le a_{2n+2} \le a_{2n+3} \le a_{2n+1}$ for all $n \ge 0$ which does not converge.

G. Fill in the details of how the Completeness Theorem implies the Least Upper Bound Principle.

H. Let $a_0 = 0$ and set $a_{n+1} = \cos(a_n)$ for $n \ge 0$. Try this on your calculator (use radian mode!).

(a) Show that $a_{2n} \le a_{2n+2} \le a_{2n+3} \le a_{2n+1}$ for all $n \ge 0$.
(b) Use the Mean Value Theorem to find an explicit number $r < 1$ such that $|a_{n+2} - a_{n+1}| \le r|a_n - a_{n+1}|$ for all $n \ge 0$. Hence show that this sequence is Cauchy.
(c) Describe the limit geometrically as the intersection point of two curves.

I. Evaluate the continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} \; .$$

J. Let $x_0 = 0$ and $x_{n+1} = \sqrt{5 - 2x_n}$ for $n \ge 0$. Show that this sequence converges and compute the limit. HINT: Show that the even terms increase and the odd terms decrease.

K. Consider an infinite binary expansion $(0.e_1 e_2 e_3 \ldots)_{\text{base } 2}$, where each $e_i \in \{0, 1\}$. Show that $a_n = \sum_{i=1}^{n} 2^{-i} e_i$ is Cauchy for every choice of zeros and ones.

L. One base-independent construction of the real numbers uses Cauchy sequences of rational numbers. This exercise asks for the definitions that go into such a proof.

(a) Find a way to decide when two Cauchy sequences should determine the same real number without using their limits. HINT: Combine the two sequences into one.
(b) Your definition in (a) should be an equivalence relation. Is it? (See Appendix 1.3.)
(c) How are addition and multiplication defined?
(d) How is the order defined?

2.9 Countable Sets

Cardinality measures the size of a set in the crudest of ways—by counting the numbers of elements. Obviously, the number of elements in a set could be 0, 1, 2, 3, 4, or some other finite number. Or a set can have infinitely many elements. Perhaps