HW #2 SOLUTION

2.4 (C) Arguing by CONTRADICTION, suppose $L > M$

Then $L - M = \delta > 0$

Since $\lim_{n\to\infty} a_n = L$, then $\exists N_1 \in \mathbb{N}$ s.t. $|a_n - L| < \frac{\delta}{3}$, $\forall n > N_1$

Since $\lim_{n\to\infty} b_n = M$, then $\exists N_2 \in \mathbb{N}$ s.t. $|b_n - M| < \frac{\delta}{3}$ $\forall n > N_2$

It follows that, if $n > \max(N_1, N_2)$, then $b_n < a_n$

In fact $a_n - b_n = (a_n - L) + (L - M) + (M - b_n) = \delta + (a_n - L) + (M - b_n) \geq \frac{\delta}{3} > 0$

CONTRADICTION. It must be $L \leq M$

2.4 (G) WLOG Suppose $\lim_{n\to\infty} a_n = L > 0$

Then is $N \in \mathbb{N}$ s.t. $|a_n - L| \leq \frac{L}{2}$ $\forall n > N$

It follows that $a_n = (a_n - L) + L \geq L - \frac{L}{2} = \frac{L}{2} > 0$ $\forall n > N$

2.5 (C) Suppose $\lim a_n = L > 0$. Assume that $a_n \geq 0$

Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$ if $n \geq N$

Note that $\sqrt{a_n} - \sqrt{L} = \frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}$. Hence $|\sqrt{a_n} - \sqrt{L}| \leq \frac{|a_n - L|}{\sqrt{L}}$

It follows that, if $n > N$, then $|\sqrt{a_n} - \sqrt{L}| < \frac{\varepsilon}{\sqrt{L}}$

Since $L > 0$, then $\varepsilon/\sqrt{L}$ can be made arbitrarily small.

This shows that ~~also~~ $\sqrt{a_n} \to \sqrt{L}$.

2.5 (D) Suppose $\lim a_n = L$ and $|a_n - b_n| < \frac{1}{n}$

Given ~~$n \in \mathbb{N}$~~ $\varepsilon > 0$, $\exists N_1$: $|a_n - L| < \varepsilon/2$ if $n > N_1$

Choose N_2: $\frac{1}{n} < \frac{\varepsilon}{2}$ if $n > N_2$. Set $N = \max\{N_1, N_2\}$

Now $|b_n - L| \leq |b_n - a_n| + |a_n - L| < \frac{1}{n} + |a_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ if $n > N$.

Hence $(b_n) \to L$

2.6 (B) claim: (a_n) is INCREASING.

Proof by INDUCTION: $a_1 = 0$, $a_2 = \sqrt{5}$ $\Rightarrow$ $a_1 \geq a_0$

Assume $a_{n+1} \geq a_n$.

Then $a_{n+2} = \sqrt{5 + 2a_{n+1}} \geq \sqrt{5 + 2a_n} = a_{n+1}$ ✓

claim (a_n) is bdd above

Proof by INDUCTION: $a_1 \leq 3$

Assume $a_n \leq 4$

$a_{n+1} = \sqrt{5 + 2a_n} \leq \sqrt{5 + 8} \leq 4$ ✓

To find L we set

$L = \lim a_{n+1} = \lim \sqrt{5 + 2a_n} = \sqrt{5 + 2L}$ $\Rightarrow$ $L^2 - 2L - 5 = 0$

(choose positive root) $L = 1 + \sqrt{6}$

2.7 H Suppose every $(x_{n_k})_k$ has a subsequence $(x_{n_{k(\ell)}})_\ell$ with $\lim_{\ell} x_{n_{k(\ell)}} = L$

Arguing by CONTRADICTION, suppose $(x_n) \not\to L$.

This implies that $\exists (x_{n_k})$ ~~s.t. $|x_{n_k} - L$~~ and a $\delta > 0$ s.t.

$$|x_{n_k} - L| > \delta \quad \forall k$$

However we know that (x_{n_k}) has a subsequence $(x_{n_{k(\ell)}}) \to L$.

This implies that, for k sufficiently large, there are values such that $|x_{n_k} - L|$ can be made arbitrarily small, i.e.

$\exists k$ s.t. $|x_{n_k} - L| < \varepsilon$ for any $\varepsilon > 0$. This is a CONTRADICTION.

It must be that $(x_n) \to L$.

2.7 J $\liminf a_n = \lim_{n\to\infty} \left(\inf_{k \ge n} a_k \right) = L$

~~Since the limit exist, then the sequence $\left(\inf_{k\ge n} a_k\right)_{n=1}^{\infty}$ is bounded.~~
~~By the Bolzano-Weierstrass theorem, every bounded sequence has a convergent subsequence.~~
~~In fact, it is easy to verify that $\left(\inf_{k\ge n} a_k\right)_{n=1}^{\infty}$ is decreasing as $\{a_k : k \ge n+1\} \supseteq \{a_k : k \ge n\}$.~~

Since $L = \liminf a_k$, $\exists a_{k_1}$ s.t. $|a_{k_1} - L| < \frac{1}{2}$. The existence of such a_{k_1} follows from the def. of lim inf

Similarly, $\exists a_{k_2}$, with $k_2 > k_1$ s.t. $|a_{k_2} - L| < (\frac{1}{2})^2$

We can repeat, showing that $\exists a_{k_n}$ with $|a_{k_n} - L| < (\frac{1}{2})^n$

Hence $\exists (a_{k_n}) \subset (a_k)$ with $(a_{k_n}) \to L$.

2.8 A Since (x_n) is Cauchy, given $\varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon/2 \quad \text{if } n, m > N_1$$

Since $(x_{n_k}) \to a$, $\exists N_2 \in \mathbb{N}$ s.t.

$$|x_{n_k} - a| < \varepsilon/2 \quad \text{if } k > N_2$$

Let $N = \max(N_1, N_2)$, then

$$|x_n - a| \le |x_n - x_{n_k}| + |x_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{if } n > N$$

2.8 B Let $a_n = \sqrt{n}$. Clearly (a_n) diverges.

$$|a_{n+1} - a_n| = |\sqrt{n} - \sqrt{n+1}| = \frac{|n - (n+1)|}{\sqrt{n} + \sqrt{n+1}} = \frac{1}{\sqrt{n} + \sqrt{n+1}} \to 0 \text{ as } n \to \infty$$