HW #3 SOLUTION

2.8 C Suppose $\lim_{N\to\infty} \sum_{n=1}^{N} |a_n - a_{n+1}| < \infty$.

let $S_M = \sum_{K=1}^{M} |a_K - a_{K+1}|$. Since (S_M) converges, then it is Cauchy

and given $\varepsilon > 0$, $\exists N$ s.t. $|S_n - S_m| < \varepsilon$ if $n, m > N$

$$|S_n - S_m| = \sum_{K=1}^{n} |a_K - a_{K+1}| - \sum_{K=1}^{m} |a_K - a_{K+1}| = \sum_{K=m+1}^{n} |a_K - a_{K+1}| < \varepsilon \text{ if } n,m > N$$

Assume $n > m$

$$|a_n - a_m| \leq |a_n - a_{n-1}| + \dots + |a_{m+1} - a_m| = \sum_{K=m+1}^{n} |a_K - a_{K+1}| < \varepsilon \text{ if } n, m > N$$

This shows that (a_n) is a Cauchy seq.

2.9 A We proved that $\exists h$ bijection $h : \mathbb{N} \to \mathbb{Z}$. Hence $|\mathbb{N} \times \dots \times \mathbb{N}| = |\mathbb{Z} \times \dots \times \mathbb{Z}|$

It follows that our proof is completed if we show that $\mathbb{N} \times \dots \times \mathbb{N}$ is countable

Proof by INDUCTION. let P_n: $\mathbb{N} \times \dots \times \mathbb{N} = \mathbb{N}^{n+1}$ is countable

(1) P_1: $\mathbb{N} \times \mathbb{N}$ is countable as proved in class $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

(2) P_n: Assume $\mathbb{N}^{n+1}$ is countable : $|\mathbb{N}^{n+1}| = |\mathbb{N}|$

(3) $\mathbb{N}^{n+2} = \mathbb{N}^{n+1} \times \mathbb{N}$.

Since $\mathbb{N}^{n+1}$ is countable, $\exists$ bijection $g : \mathbb{N} \to \mathbb{N}^{n+1}$

Define $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}^{n+1} \times \mathbb{N}$ by $F(n,m) = (g(n), m)$

This map is a bijection. Hence $|\mathbb{N}^{n+1} \times \mathbb{N}| = |\mathbb{N}^{n+2}| = |\mathbb{N}|$

2.9 C Suppose $|A| \leq |B|$, $|B| \leq |C|$. Hence $\exists$ injections $f : A \to B$, $g : B \to C$.

claim $g \circ f : A \to C$ is injective. In fact, if $a_1, a_2 \in A$, $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$

Since g is also injective, $g(f(a_1)) \neq g(f(a_2))$. Hence $g \circ f$ is INJECTIVE

It follows that $|A| \leq |C|$.

3.1 A $\frac{1}{n(n+2)} = \frac{1/2}{n} - \frac{1/2}{n+2}$.

$$\sum_{K=1}^{n} \frac{1}{K(K+2)} = \frac{1}{2} \sum_{K=1}^{n} \left(\frac{1}{K} - \frac{1}{K+2}\right) = \frac{1}{2}\left(1 - \cancel{\tfrac{1}{3}} + \frac{1}{2} - \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}} + \dots\right.$$

$$\left. + \dots \cancel{\tfrac{1}{n-2}} - \cancel{\tfrac{1}{n}} + \cancel{\tfrac{1}{n-1}} - \frac{1}{n+1} + \cancel{\tfrac{1}{n}} - \frac{1}{n+2}\right) =$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) \to \boxed{\frac{3}{4}}$$

3.1 C If $t_K \geq 0$ and $\sum t_K$ is convergent then ~~it must be $|t_K| < 1$ for~~ This follows from the property that $\lim_{K\to\infty} t_K = 0$

then must be N s.t. $t_K < 1$ if $K > N$.

Write $\sum_{K=1}^{\infty} t_K = \sum_{K=1}^{N} t_K + \sum_{K=N+1}^{\infty} t_K$

If $p > 1$, $\sum_{K=N+1}^{\infty} t_K^p < \sum_{K=N+1}^{\infty} t_K$ since $t_K < 1$ for $K > N$

Hence $\sum_{K=1}^{\infty} t_K^p < \infty$. Note that $\sum_{K=1}^{N} t_K^p$ is a finite sum and it must be bounded since all t_K are bounded.