HW 4 SOLUTION

3.2F Since $(a-b)^2 \ge 0$, then $a^2+b^2 \ge 2ab$

$$2\sum_k a_k b_k \le \sum_k (a_k^2+b_k^2) < \infty$$

Hence $\sum_k a_k b_k$ converges

3.2G Let $(a_k) = \left(\frac{(-1)^k}{\sqrt{k}}\right)$, $(b_k) = \left(\frac{(-1)^k}{\sqrt{k}}\right)$

Then $\sum a_k$, $\sum b_k$ converge but $\sum a_k b_k = \sum \frac{1}{k}$ diverges

3.2H - Suppose $\sum a_n$ converges

Since $\frac{a_n}{a_n+1} \le a_n$, then $\sum \frac{a_n}{a_n+1}$ also converges

- Suppose that $\sum \frac{a_n}{a_n+1}$ converges

It follows that $\lim a_n = 0$. If not $\lim_{n\to\infty} \frac{a_n}{a_n+1} \ne 0$ and the series would not converge.

Hence $\exists N$ s.t. $a_n < 1$. This IMPLIES THAT

$a_n+1 < 2$ for $n > N$.

Hence $\frac{1}{2}\sum_{n>N} a_n < \sum_{n>N} \frac{a_n}{a_n+1} < \infty$

This IMPLIES that $\sum a_n < \infty$.

3.2N Since $f(x)$ is positive and MONOTONE DECR. on $[1,\infty)$, then ~~f $x \in [k,k+1]$~~ $x \in (k-1,k]$

$$f(k) \le f(x) < f(k+1)$$

Hence $\sum_{n=2}^{k} f(n+1) \le \int_1^{k+1} f(x)\,dx \le \sum_{n=1}^{k} f(n)$

$$\sum_{n=2}^{k+1} f(n) \le \int_1^{k+1} f(x)\,dx \le \sum_{n=1}^{k} f(n)$$

3.2 N Since f is positive and MONOTONE DECR. on $[1,\infty)$, then

$$f(k+1) \le f(x) \le f(k) \quad \text{for } x \in [k,k+1]$$

then

$$\int_k^{k+1} f(k+1)\,dx \le \int_k^{k+1} f(x)\,dx \le \int_k^{k+1} f(k)\,dx$$

$$\Rightarrow \sum_{k=1}^{n} f(k+1) \le \int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} f(k)$$

$$= \sum_{k=2}^{n+1} f(k)$$

Taking the limit for $n\to\infty$, this IMPLIES that $\sum_{n=1}^{\infty} f(k) < \infty \iff \int_1^{\infty} f(x)\,dx < \infty$

4.1 D $\|x\| \le \|x-y\| + \|y\|$ and $\|y\| \le \|x-y\| + \|x\|$

then $|\|x\| - \|y\|| \le \|x-y\|$

4.1 E Let P_n: $\|x_1 + \dots + x_{n+1}\| \le \|x_1\| + \dots \|x_{n+1}\|$

P_1: $\|x_1 + x_2\| \le \|x_1\| + \|x_2\|$ TRUE BY TRIANGLE INEQ.

Assume P_k

P_{k+1}: $\|x_1 + \dots + x_{k+2}\| \le \|x_1 + \dots + x_{k+1}\| + \|x_{k+2}\|$ by triangle INEQ.

$\le \|x_1\| + \dots + \|x_{k+1}\| + \|x_{k+2}\|$ by INDUCTIVE STEP

4.1.F Suppose $\|x\| = 1$, $\|y\| = 1$.

$\left\|\frac{x+y}{2}\right\| = 1 \iff \|x+y\|^2 = 4 \iff \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle = 4$

$\iff \langle x,y\rangle = 1 \iff \langle x,y\rangle = \|x\|\,\|y\| \iff x = cy$

Since $1 = \|x\| = \|cy\| = |c|\|y\|$, then $|c| = 1$. Since $\langle x,y\rangle \ge 0$, then $c = 1$.

4.1.I Suppose $\|Ux\| = \|x\|$ $\forall x \in \mathbb{R}^n$

Then $\langle Ux + Uy, Ux + Uy\rangle = \langle Ux, Ux\rangle + \langle Uy, Uy\rangle + 2\langle Ux, Uy\rangle$

$= \|x\|^2 + \|y\|^2 + 2\langle Ux, Uy\rangle$

$\|$

$\langle U(x+y), U(x+y)\rangle = \|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle$.

Hence $\langle Ux, Uy\rangle = \langle x,y\rangle$ $\forall x,y \in \mathbb{R}^n$