HW5 SOLUTION

4.2 A (a) Let $(x_k) \subset \mathbb{R}^n$ and suppose $\lim x_k = a$.

This implies that $\lim \|x_k - a\| = 0$

Since $| \|x_k\| - \|a\| | \le \|x_k - a\|$.

then $\lim \|x_k\| = \|a\|$.

(b) Converse is False. Let $x_k = ((-1)^k, \dots, (-1)^k)$.

$\lim x_k$ does not exist, but $\lim \|x_k\| = \|(1, \dots, 1)\|$.

4.2 B Suppose $\sum_n \|x_n - x_{n+1}\| < \infty$.

Let $S_n = \sum_{k=1}^{n} \|x_k - x_{k+1}\|$. Since $\lim S_n$ exists, then $\lim_k \|x_k - x_{k+1}\| = 0$

Also, (S_n) is Cauchy, hence given $\varepsilon > 0$, $\exists N$ s.t. $|S_n - S_m| < \varepsilon$ if $n, m > N$

Assume $n > m$. $S_n - S_m = \sum_{k=1}^{n} \|x_k - x_{k+1}\| + \sum_{k=1}^{m} \|x_k - x_{k+1}\|$

$= \sum_{k=m+1}^{n} \|x_k - x_{k+1}\| < \varepsilon$ if $n, m > N$

$\|x_n - x_m\| \le \|x_n - x_{n-1}\| + \dots + \|x_{m+1} - x_m\| = \sum_{k=m}^{n-1} \|x_k - x_{k+1}\|$

This shows that (x_n) is Cauchy.

4.2. E Let $M \subset \mathbb{R}^n$ and $\{v_1, \dots, v_m\}$ be an ONB of M

(a) A sequence $(x_k) \subset M$ converges to $a = \sum_{j=1}^{m} \alpha_j v_j$ iff $\lim \langle x_k, v_j \rangle = \alpha_j$ $1 \le j \le m$

Proof Any $x_k \in M$ can be written as

$$x_k = \sum_{j=1}^{m} \langle x_k, v_j \rangle v_j$$

since $\{v_1, \dots, v_m\}$ is an ONB of M

- Suppose $\lim x_k = a$.

then $\lim \|x_k - a\|^2 = \lim \| \sum (\langle x_k, v_j \rangle - \langle a, v_j \rangle) v_j \|^2 =$

$= \lim \sum_j | \langle x_k, v_j \rangle - \langle a, v_j \rangle |^2 = \lim \sum_j | \langle x_k, v_j \rangle - \alpha_j |^2$

then $\lim \langle x_k, v_j \rangle = \alpha_j$ $1 \le j \le n$.

- Conversely, suppose $\lim \langle x_k, v_j \rangle = \alpha_j$, $1 \le j \le n$

By the same sequence of equalities above,

$\lim \|x_k - a\| = 0$.

then $\lim x_k = a$.

(b) Let $(x_k) \subset M$ be a Cauchy seq.

It follows that, given $\varepsilon > 0$, $\exists N$ s.t. $\|x_\ell - x_n\| < \varepsilon$ if $\ell, n > N$

Since $(x_k) \subset M$, we can write $x_k = \sum_{j=1}^{m} \langle x_k, v_j \rangle v_j$

As in the calculation in part (a),

$$\|x_\ell - x_n\|^2 = \sum_{j=1}^{m} |\langle x_\ell, v_j \rangle - \langle x_n, v_j \rangle|^2 < \varepsilon^2 \quad \text{if } \ell, n > N$$

This shows that the NUMERICAL SEQUENCES $(\langle x_\ell, v_j \rangle)_{\ell=1}^{\infty}$ are Cauchy for any $1 \le j \le m$. Hence $\langle x_\ell, v_j \rangle \to \langle y, v_j \rangle$ $\quad 1 \le j \le m$

Thus, letting $y = \sum_{j=1}^{m} \langle y, v_j \rangle v_j \in M$, we conclude

that $\lim x_k = y \in M$. This shows that M is COMPLETE.

4.3 B Let $(a_n) \subset \mathbb{R}^k$ with $\lim a_n = a$.

a is clearly a limit point of $S = \{a_n : n \ge 1\} \cup \{a\}$

For any $a_n \in S$, the seq. $(a_n, a_n, \ldots)$ has limit point $a_n \in S$.

Hence S contains all its limit points and is closed.

4.3 G As shown in 4.3 L, Q and hence Q^n have empty INTERIOR but are NOT CLOSED

4.3 J Suppose U open, A closed.

Let $S = U \setminus A = U \cap A^c$. $\quad S^c = U^c \cup A$. This is the union of two closed sets and must be closed.

• $A \setminus U$ can be neither open nor closed. Eg. $A = [0,1]$, $U = (-\frac{1}{2}, \frac{1}{2})$, $A \setminus U = (\frac{1}{2}, 1]$.

4.3 L (a) Let Q = rationals, I = irrational, $\mathbb{R}$ real.

Clearly $I \subset \mathbb{R}$. Need to show that $\bar{I} \supset \mathbb{R}$.

By the Archimedean property, given any $x \in \mathbb{R}$, we can find a sequence of irrational numbers converge to x. Hence x is a limit point of I, $x \in \bar{I}$.

This shows that $\mathbb{R} \subset \bar{I}$

(b) By the Archimedean property, given any $q \in Q$, for any $r > 0$, the ball $B_r(q)$ contains points belonging to I.

Hence Q has empty INTERIOR