HW #6 SOLUTION

4.4 C Let $(A_i)_{i=1}^{n}$ be compact sets in $\mathbb{R}^n$

Each set A_i is closed and bounded. Since the union of finitely many closed sets is closed, and the union of finitely many bounded sets is bounded, then $\bigcup_{i=1}^{n} A_i$ is closed and bounded, hence it is compact.

4.4 D Let $\{A_i : i \in I\}$ be a countable collection of compact sets in $\mathbb{R}^n$. Let $S = \bigcap_{i \in I} A_i$.

Since $S \subset A_1$ and A_1 is bounded, then S is bounded.

Since each A_i is closed, then S is also closed.

Then S is compact.

4.4 F Let $(x_n) \subset \overline{K} \subset \mathbb{R}^n$, where $\overline{K}$ is compact.

Suppose (x_n) is NOT convergent. Since $\overline{K}$ is compact, then $\exists$ subseq. $(x_{n_k}) \to x \in \overline{K}$.

Since (x_n) is NOT convergent, we can find ∞-tely many terms of the seq. away from x, ~~Hence~~, given $\delta > 0$, we can find a subseq. (x_{n_j}) s.t. $d(x_{n_j}, x) > \delta$ $\forall n_j$.

Since $\overline{K}$ is compact, we can find a convergent subsequence $(x_{n_{j_\ell}})$ of (x_{n_j}) which converges to some y, with $d(x, y) > \delta > 0$.

4.4 I Let A, B be closed sets with $A \cap B = \emptyset$.

Let $d(A,B) = \inf\{\|a-b\| : a \in A, b \in B\}$.

(a) ~~Let $\{a\}$ be a singleton. Hence, there is $\varepsilon > 0$, s.t. $B_\varepsilon(a) \cap B = \emptyset$~~

~~This implies that $\|a-b\| \geq \varepsilon$, for all $b \in B$.~~

~~Hence $d(a,B) \geq 0$.~~

(b) Suppose A is compact.

Arguing by contradiction, suppose that there is a sequence $(a_n) \subset A$ and $(b_n) \subset B$ s.t. $\lim \|a_n - b_n\| = 0$.

Since A is compact, (a_n) has a convergent subsequence in A, say $(a_{n_k}) \to a \in A$.

Hence $\|a - b_{n_k}\| \leq \|a - a_{n_k}\| + \|a_{n_k} - b_{n_k}\| \to 0$

This implies that a is a limit point of B.

This cannot be since B is closed.

(c) Let $A = \{(x,y) : x \geq 0, y \leq 0\}$, $B = \{(x,y) : x \geq 0, y \geq \frac{1}{x}\}$

A and B are closed.

Let $a_n = (n, 0)$, $b_n = (n, \frac{1}{n})$. $a_n \in A$, $b_n \in B$

$$\lim_{n\to\infty} \|a_n - b_n\| = \lim_{n \to \infty} \|(n,0) - (n, \tfrac{1}{n})\| = 0$$

This shows that $d(A,B) = 0$.

(a) Arguing by contradiction, suppose $d(a,B) = 0$.

Hence there exists a sequence $(b_n) \subset B$ s.t. $\lim b_n = a$.

This implies that a is a limit point of B.

Since B is closed, it must contain all its limit points, implying that $a \in B$. This contradicts the assumption that a, B are disjoint. Hence it must be $d(a,B) > 0$.