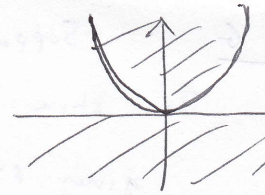
HW 7 SOLUTION



5.1 E (a)
$$f(x,y)=\begin{cases}0 & \text{if } y\le 0,\ y\ge x^2\\ \sin\left(\frac{\pi x^2}{y}\right) & \text{if } 0<y<x^2\end{cases}$$

Set $y=\frac{3}{4}x^2 \Rightarrow f(x,y(x))=\sin\left(\frac{4}{3}\pi\right)$

Hence $\lim\limits_{x\to 0} f(x,y(x))=\sin\left(\frac{4}{3}\pi\right)\neq f(0,0)=0$

(b) Set $y=\alpha x$, $\alpha\neq 0$. then
$$f(x,\alpha x)=\begin{cases}0 & \text{if } y\le 0,\ y\ge x^2\\ \sin\frac{\pi}{\alpha}x & \text{if } 0<y<x^2\end{cases}$$

$$\lim_{x\to 0} f(x,\alpha x)=0=f(0,0)$$

5.1 F (a) $\lim\limits_{x\to a} f(x)=v$ $\iff$ For every $\varepsilon>0$, there is an $r>0$ s.t. $\|f(x)-v\|<\varepsilon$ whenever $0<\|x-a\|<r$, $x\in S$

$\iff$ For every $\varepsilon>0$, there is an $r>0$ s.t. $f(x)\in B_\varepsilon(v)$ whenever $x\in B_r(a)\cap S$

$\iff$ For every $\varepsilon>0$, there is an $r>0$ s.t. $f(B_r(a)\cap S\setminus\{a\})\subset B_\varepsilon(v)$

(b) f continuous at $a\in S$ $\iff$ For every $\varepsilon>0$, there is an $r>0$ s.t. $\|f(x)-f(a)\|<\varepsilon$ whenever $\|x-a\|<r$, $x\in S$

$\iff$ For every $\varepsilon>0$, there is an $r>0$ s.t. $f(x)\in B_\varepsilon(f(a))$ whenever $x\in B_r(a)\cap S$

$\iff$ For every $\varepsilon>0$, there is an $r>0$ s.t. $f(B_r(a)\cap S)\subset B_\varepsilon(f(a))$

5.1 J Suppose $f:[0,5]\to\mathbb{R}$ is differentiable with $|f'(x)|\le M$ $\forall x\in[0,5]$

By MVT, for every $x,y\in[0,5]$, $f(x)-f(y)=f'(c)(x-y)$, where $c\in(x,y)$

Hence $|f(x)-f(y)|\le M|x-y|$ $\forall x,y\in[0,5]$

5.2 F

- Suppose $\lim\limits_{x\to a} f(x)=L$.
  By def., given $\varepsilon>0$, there exists an $r>0$ s.t. $|f(x)-L|<\varepsilon$ whenever $|x-a|<r$
  This implies that $|f(x)-L|<\varepsilon$ whenever $a-r<x<a+r$
  Hence $|f(x)-L|<\varepsilon$ whenever $a-r<x<a$ and $a<x<a+r$
  This implies that $\lim\limits_{x\to a^-} f(x)=L$ and $\lim\limits_{x\to a^+} f(x)=L$
- Conversely, suppose $\lim\limits_{x\to a^-} f(x)=L$ and $\lim\limits_{x\to a^+} f(x)=L$
  By def., given $\varepsilon>0$, there exists $r>0$ s.t. $|f(x)-L|<\varepsilon$ whenever $a-r<x<a$ and $a<x<a+r$
  Hence, given $\varepsilon>0$, there is $r>0$ s.t. $|f(x)-L|<\varepsilon$ whenever $a-r<x<a+r$ $\iff$ $|x-a|<r$
  This implies that $\lim\limits_{x\to a} f(x)=L$.

5.2 G

Suppose f is INCREASING on $[a,b]$ and $|f(x)| < L = \sup_{x\in(a,b)} f(x)$

To show that $\lim_{x\to b^-} f(x)$ exists, we need to show that given $\varepsilon>0$, there exist an $r>0$ s.t. $|f(x) - L| < \varepsilon$ whenever $b-r<x<b$.

Since L is an upper bound of $f(x)$ clearly $f(x) < L+\varepsilon$, and $L-\varepsilon$ is NOT an upper bound. Hence, there exists a y such that $f(y) > L-\varepsilon$.

Since f is INCREASING, $f(x) > f(y) > L-\varepsilon$ if ~~$x>y>b$~~ $b>x>y$

Choose $r = b-y$. Then $|f(x)-L|<\varepsilon$ if $\underbrace{b-r}_{= y} < x < b$.

5.2 H

$$f(x) = x\chi_{\mathbb{Q}}(x), \quad x\in\mathbb{R}$$

$$f(a) = \begin{cases} a & \text{if } a\in\mathbb{Q} \\ 0 & \text{if } a\notin\mathbb{Q}\end{cases}$$

claim $\lim_{x\to a} f(x) =$ DNE for any $a\in\mathbb{R}\setminus\{0\}$

In fact, let $a\neq 0$. By the Archimedean principle, given any $a\in\mathbb{Q}$ we can find $x\notin\mathbb{Q}$ s.t. $|f(x)-f(a)| = |a|$ where $|x-a|$ can be made arbitrarily small.

Similarly, given any $a\notin\mathbb{Q}$, we can find $x\in\mathbb{Q}$ s.t. $|f(x)-f(a)| = |x|$ where $|x-a|$ can be made arbitrarily small.

Set $|x-a| < |a|/2$. Then $|f(x)-f(a)| > |a|/2$ for any $|x-a| <$

This shows that $\lim_{x\to a} f(x) =$ DNE if $a\neq 0$.

If $a=0$, then $\lim_{x\to 0}|f(x)| \le \lim_{x\to 0}|x| = 0$.

Thus $f(x)$ is continuous at $x=0$.