HW # 8 SOLUTION

 # 5.3 F Let f, g be continuous functions from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$.

Let $h(x) = \langle f(x), g(x) \rangle$.

To prove that h is continuous at $x \in S$, let $(x_n) \subset S$ and suppose that $\lim x_n = x$.

Since f, g are continuous at x, $\lim_{n\to\infty} f(x_n) = f(x)$, $\lim_{n\to\infty} g(x_n) = g(x)$. Also $\|f(x)\|, \|g(x_n)\|$ are bounded.

By Cauchy-Schwarz ineq.

$$|h(x) - h(x_n)| = |\langle f(x), g(x)\rangle - \langle f(x_n), g(x_n)\rangle| \le |\langle f(x), g(x)\rangle - \langle f(x), g(x_n)\rangle| +$$
$$+ |\langle f(x), g(x_n)\rangle - \langle f(x_n), g(x_n)\rangle| \le \|f(x)\|\, \|g(x) - g(x_n)\| + \|f(x) - f(x_n)\|\, \|g(x_n)\|$$

This shows that $\lim_{n\to\infty} h(x_n) = h(x)$. Hence $h(x)$ is continuous on S.

 # 5.3 I Let $A = \{(x,y) : x \in U,\ y > f(x)\}$ where $U \subset \mathbb{R}^n$ is open and $f: U \to \mathbb{R}$ is continuous.

Since U is open, for any $x \in U$, there is an $r > 0$ s.t. $B_r(x) \subset U$.

Fix $\varepsilon > 0$. Pick $a \in U$ and $y > f(a) + \varepsilon$.

Since f is continuous, there is an r s.t. $\|x - a\| < r$ implies $|f(x) - f(a)| < \varepsilon$.

This implies that $y > f(x)$ for all $x \in B_r(a)$.

It follows that, for any $(a,y) \in A$, there is a ball $B_\rho(a,y) \subset A \subset \mathbb{R}^{n+1}$. Here $\rho = \min(\varepsilon, r)$.

 # 5.3 K Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. Let $B \subset \mathbb{R}^m$ be closed. Then B^c is open and $f^{-1}(B^c)$ is open. However $f^{-1}(B^c) = (f^{-1}(B))^c$, hence $f^{-1}(B)$ is closed. This shows that the INVERSE IMAGE of any closed set is closed.

Conversely, suppose that for any $B \subset \mathbb{R}^m$ closed, $f^{-1}(B)$ is closed. Let $U \subset \mathbb{R}^m$ be open. Then U^c is closed and $f^{-1}(U^c)$ is closed. Then $f^{-1}(U)$ is open and f must be continuous by result proved in class.

 # 5.3 N Suppose $f(u+v) = f(u) + f(v)$ $\forall u, v \in \mathbb{R}$

(a) $P(m)$: $f(mx) = mf(x)$ $\forall x \in \mathbb{R}$, $m \in \mathbb{N}$.

$m = 1$: $f(x) = f(x)$ ✓

Assume $f(mx) = m f(x)$

$f((m+1)x) = f(mx) + f(x)$ by def.

$= m f(x) + f(x) = (m+1) f(x)$ by INDUCTIVE STEP

(b) $f\left(\frac{p}{q}\right) = p f\left(\frac{1}{q}\right) = \frac{p}{q} \cdot q f\left(\frac{1}{q}\right) = \frac{p}{q} f\left(\frac{q}{q}\right) = \frac{p}{q} f(1)$, $p, q \in \mathbb{N}$, $q \neq 0$

(c) Suppose f continuous on $\mathbb{R}$. We have $f(x) = f(1)\, x$ for $x \in \mathbb{Q}$

Let $x \in \mathbb{R}$. There is a sequence $(x_n) \subset \mathbb{Q}$ s.t. $\lim x_n = x$.

For any x_n, $f(x_n) = f(1)\, x_n$. Since f continuous, $\lim_{n\to\infty} f(x_n) = f(x)$, $\lim_{n\to\infty} x_n = x$.

This shows that it must be $f(x) = f(1)\, x$, for all $x \in \mathbb{R}$.

 # 5.4 H f: periodic and continuous. $f(x) = f(x+d)$, $x \in \mathbb{R}$, $d > 0$.

By Extreme value theorem, $f(x)$ is UNIF. CONTINUOUS on $[0,d]$, hence $\exists\, a, b$ s.t. $m = \min_{[0,d]} f(x) = f(a)$, $a, b \in [0,d]$

and $M = \max_{[0,d]} f(x) = f(b)$.

For any $x \in \mathbb{R}$, there is n s.t. $x = nd + y$, $y \in [0,d]$ and $f(x) = f(nd + y) = f(y)$.

Hence $f(x) \ge m$ $\forall x \in \mathbb{R}$, $f(x) \le M$ $\forall x \in \mathbb{R}$.

#5.4 E $f: S \subset \mathbb{R}^n \to \mathbb{R}^m$. S compact. $G(f) = \{(x, f(x)) : x \in S\}$

- Suppose G is compact. Arguing by contradiction, suppose f is NOT CONTINUOUS. Then there is a sequence $(x_n) \subset S$ with $\lim x_n = x \in S$ but $\lim f(x_n) \neq f(x)$.
  The sequence $(x_n, f(x_n)) \subset G(f)$. Since this set is compact, there is a subsequence $(x_{n_k}, f(x_{n_k}))$ converging to a point $(a,b) \in G(f)$.
  Since (x_n) converges to x, then $a = x$. However since f is NOT continuous, $b \neq f(x)$.
  Hence we found that $(x_{n_k}, f(x_{n_k})) \to (x, b)$ with $b \neq f(x)$, that is, $(x,b) \notin G(f)$.
  This is a contradiction since G is compact. Hence f must be continuous.
- Suppose f is continuous. Let $(x_n) \subset S$. Since S is compact, there is a subsequence (x_{n_k}) of (x_n) such that $\lim x_{n_k} = x \in S$.
  By the continuity of f, $\lim f(x_{n_k}) = f(x)$.
  Hence we have proved that for any sequence $(x_n, f(x_n)) \subset G(f)$, there is a subsequence $(x_{n_k}, f(x_{n_k}))$ converging to a point $(x, f(x))$ in $G(f)$. Hence $G(f)$ is compact.

#5.4 L Let $f: C \subset \mathbb{R}^n \to D \subset \mathbb{R}^m$. Suppose f is continuous and one-to-one.
Since f is one-to-one, then f^{-1} exists.
Let $B \subset C$ be closed. Since C is compact, then B is compact and $f(B)$ is compact, hence closed, in D. Now $(f^{-1})^{-1}(B) = f(B)$.
This shows that the INVERSE IMAGE of any closed set B under f^{-1} is a closed set, hence f^{-1} is CONTINUOUS. [By #5.3 K]

[Alternate Proof] Arguing by contradiction, suppose f^{-1} is NOT continuous.
Hence there is $a \in C$ and $\varepsilon > 0$ s.t. for each $r > 0$ there is $x \in S$ s.t.

$$\|f(x) - f(a)\| < r \quad \text{and} \quad \|a - x\| > \varepsilon$$

Let $r_k = \frac{1}{k}$. We can find a sequence $(x_k) \subset C$ s.t.

$$\|f(x) - f(x_k)\| < \tfrac{1}{k} \quad \text{while} \quad \|x - x_k\| \geq \varepsilon$$

Since C is compact, there is a subsequence $(x_{n_k}) \subset (x_n)$ with $(x_{n_k}) \to b \in S$.
Since f is continuous, $f(x_{n_k}) \to f(b)$.
Since $(f(x_n)) \to f(a)$, then $f(b) = f(a)$. However $\|x - a\| \geq \varepsilon > 0$.
This contradicts the hypothesis that f is one-to-one.

#5.5 F Let f periodic and continuous on $\mathbb{R}$. Let $f(x) = f(x+d)$, $d > 0$.
$f(x)$ is unif continuous on $[0,d]$. As already observed in 5.4 H, f achieves max on $[0,d]$ and the same max is the max for f on $\mathbb{R}$.
Let $x \in \mathbb{R}$. There exists an $m \in \mathbb{Z}$ s.t. $f(x) = f(x - md)$ and $y = x - md \in [0,d]$.
Hence if f is continuous on $[0,d]$, then it is continuous for all $x \in \mathbb{R}$.
In fact: given $\varepsilon > 0$, $|f(x) - f(a)| = |f(x-md) - f(a-md)| < \varepsilon$ provided $|x - a| = |(x-md) - (a-md)| < r$.

That is, the continuity at $a \in \mathbb{R}$ can be satisfied by checking a corresponding point in the interval $[0,d]$.