# HOMEWORK # 9

**5.5 I** ($\Leftarrow$) Assume $\lim_{x\to 0^+} f(x) = L$.

Hence, given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(x) - L| < \varepsilon/2$ if $0 < x < \delta$

Hence $|f(x) - f(y)| \le |f(x) - L| + |L - f(y)| < \varepsilon$ if $x, y \in (0, \delta]$

If $x \in [\delta, 1]$, $f$ is unif continuous since the domain is a closed and bounded interval. Thus $f$ is uniformly continuous

($\Rightarrow$) Assume $f$ is unif continuous on $(0,1]$. Arguing by contradiction, suppose that $\lim_{x\to 0^+} f(x)$ does not exist. Hence, we can find sequences $(x_n) \to 0$, $(y_n) \to 0$, such that $\lim |f(x_n) - f(y_n)| > \varepsilon$ for some $\varepsilon > 0$.

This implies that $f$ is not uniformly continuous. Contradiction.

**5.5.K** (a) Suppose $|f(x_1) - f(x_2)| \le M\,|x_1 - x_2|^{\alpha}$, $\alpha > 0$

Given $\varepsilon > 0$, let $r = (\varepsilon/M)^{1/\alpha}$. If $|x_1 - x_2| < r$, then

$$|f(x_1) - f(x_2)| \le M\left(\frac{\varepsilon}{M}\right)^{\frac{\alpha}{\alpha}} = \varepsilon .$$ Hence $f$ is continuous.

(b) Since $\alpha > 1$, we can write $\alpha = 1 + \delta$, $\delta > 0$.

$$|f(x) - f(y)| \le M\,|x-y|^{\alpha} = M\,|x-y|\,|x-y|^{\delta}$$

Hence $\dfrac{|f(x) - f(y)|}{|x-y|} \le M\,|x-y|^{\alpha}$. This implies that $\lim_{x\to y} \dfrac{|f(x)-f(y)|}{|x-y|} = 0$.

This shows that $f$ is differentiable and $f'(x) = 0$. Hence $f$ is constant

(c) Suppose $x, y \ge 0$. Let $\alpha \in (0,1)$.

Consider $f(z) = (z-1)^{\alpha} - (z^{\alpha} - 1)$, for $z \ge 1$

$$f'(z) = \alpha(z-1)^{\alpha-1} - \alpha z^{\alpha-1} \geq 0 \iff (z-1)^{\alpha-1} \geq z^{\alpha-1} \iff z^{1-\alpha} \geq (z-1)^{1-\alpha}$$

This shows that $f'(z) \geq 0$ if $z \ge 1$.

Hence $\underline{f(z) \ge f(1) = 0}$ for $z \ge 1$.

Let $z = \dfrac{x}{y}$, where $x \ge y$. Then we have that

$$\left(\frac{x}{y}\right)^{\alpha} - 1 \le \left(\frac{x}{y} - 1\right)^{\alpha} \Rightarrow \boxed{x^{\alpha} - y^{\alpha} \le (x-y)^{\alpha}}$$

**5.6 D** If $p(x)$ is a polynomial of odd degree, then

$$\lim_{x\to\infty} p(x) = \infty, \quad \lim_{x\to-\infty} p(x) = -\infty.$$

By IVT, there should be at least one point $x_0$ where $p(x_0) = 0$

**5.6 F** Let $f: S \to \mathbb{R}$ when $S$ is a circle and $f$ is continuous.

Let $\gamma(t)$; $t \in [0,1]$ be a parametrization of the circle.

Since $f$ is continuous, then $\lim_{t\to 0} f(\gamma(t)) = \lim_{t \to 1} f(\gamma(t)) = L$

If $f$ is constant, then it is NOT one-to-one and the proof is complete.

If $f$ is not constant, then $f$ must achieve a max and a min, at some location $x_0 = \gamma(t_0)$. WLOG, assume $f(\gamma(t_0)) \geq M > L$

By IVT $f$ must assume all values in $(L, M]$ for $t \in [0, t_0]$ and $(t_0, 1]$.

Hence $f$ is NOT ONE-to-ONE.

**5.6 H(a)** Suppose $f$ is continuous on $\mathbb{R}$ and takes every value EXACTLY TWICE.

Here, there are exactly 2 points $x_1, x_2$ when $f(x_1) = f(x_2) = 0$.

By EXTREME VALUE THM., $f$ has a loc. MAX and loc. MIN in $[x_1, x_2]$.

Note that $f$ cannot be constant in $[x_1, x_2]$, since it has only 2 zeros.

WLOG, suppose $f$ has a local MAX at $c \in [x_1, x_2]$.

Note that it must be $f(x) < 0$ for $x < x_1$, $x > x_2$.

Hence there are no values of $x$ for which $f(x) < f(c)$. CONTRADICTION.

There cannot be any function as assumed in the problem.

**5.7 A**

(a) $x_1 < x_2 \Rightarrow g(x_1) \geq g(x_2) \Rightarrow f(g(x_1)) \geq f(g(x_2))$.

(b) $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$, $g(x_1) \geq g(x_2) \Rightarrow f(x_1) + g(x_1) \geq f(x_2) + g(x_2)$

(c) $x_1 < x_2 \Rightarrow f(x_1) g(x_1) \geq f(x_2) g(x_1) \geq f(x_2) g(x_2)$

**5.7 B** Suppose $f: [0,1] \to \mathbb{R}$ continuous and one-to-one.

For any ~~$x_1, x_2 \in [0,1]$~~ $0 \leq x_1 < x_2 \leq 1$, it must be $f(x_1) \neq f(x_2)$.

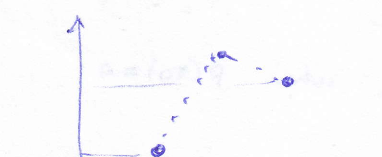
Assume $f(x_1) < f(x_2)$. ~~By IVT f takes~~

Arguing by contradiction, suppose $\exists\, c \in (x_1, x_2)$ s.t. $f(c) > f(x_2)$ or $f(c) < f(x_1)$. In the first case

By IVT, both situations would violate the INJECTIVITY. $f$ would take all values between $f(x_1)$ and $f(c)$ for $x$ in $[x_1, c]$ and all values between $f(c)$ and $f(x_2)$ for $x$ in $[c, x_2]$, so that there would be two values where $f(x) = m \in [f(c), f(x_2))$.

Similarly for the other case.

This IMPLIES that for any $x$ s.t. $x_1 < x < x_2$, it must be $f(x_1) < f(x) < f(x_2)$.




The argument is exactly the same if we assume that $f(x_1) > f(x_2)$.

This shows that $f$ must be strictly INCREASING or decreasing.