Math 4331/6312 – Fall 2016 Name:

Test #1

This is a closed book test. Please, write clearly and justify all your steps, to get proper credit for your work. You can cite general theorems from the book if needed.

(1) [3Pts] Use the definition of supremum to prove that $\sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$. Is 1 also a maximum of the set?

(2) [5 Pts] Solve the following problems about limits.

(a) State the definition of *convergence* for a sequence of real numbers.

(b) Prove that if (a_n) converges to L, then $(|a_n|)$ converges to $|L|$.

(c) Prove that if $\lim_{n\to\infty} a_n = 0$ and (b_n) is a bounded sequence, then $\lim_{n\to\infty} a_n b_n = 0$

(3)[6 Pts] For each of the following statements, prove it (you can use the theorems discussed in class) or give a counterexample (in this case, you need to explain how your counterexample disproves the statement).

(a) Every monotone sequence diverges.
(b) Every monotone sequence has a bounded subsequence.
(c) If (a_n) is a bounded sequence and (b_n) is a convergent sequence, then $(a_n b_n)$ converges.
(d) If (a_n) diverges to $+\infty$ and b_n is bounded, then $(a_n b_n)$ diverges.
(e) If (a_n) diverges to $+\infty$, b_n is bounded, and $b_n \geq 0$, then $(a_n b_n)$ diverges to $+\infty$.
(f) If (a_n) is a Cauchy sequence, then (a_n) is bounded.

(4)[4 Pts] Consider the sequence defined by

$$s_1 = 2 \text{ and } s_{n+1} = \frac{1}{4}(2s_n - 1) \text{ for } n \in \mathbb{N}.$$

(a) Prove that (s_n) is convergent (Hint: show that the sequence is monotone and bounded).
(b) Find the limit of (s_n).

TEST #1 SOLUTION

(1) $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. $1 - \frac{1}{n} \le 1 \ \forall n$ hence 1 is an upper bound

if $0 < M < 1$, then $1 - M = \delta > 0$. We can find $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \delta$.

thus $1 > 1 - \frac{1}{n} > 1 - \delta = M$. This shows that M cannot be an upper ~~bdd~~ bound.

This shows that 1 is the LEAST UPPER BOUND of S.

Since $1 \notin S$, it is NOT A MINIMUM.

(2) (a) $\lim_{n\to\infty} a_n = L$ if given any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$ if $n > N$.

(b) Suppose $\lim a_n = L$. Hence, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|a_n - L| < \varepsilon \quad \forall n > N$$

we have that $||a_n| - |L|| \le ||a_n - L| + |L| - |L|| = |a_n - L|$

Hence $||a_n| - |L|| < \varepsilon$ if $n > N$.

This proves that $\lim |a_n| = |L|$.

(c) Suppose $\lim a_n = 0$. ~~This implies that, given~~

Since (b_n) is bounded, suppose $|b_n| \le B \in \mathbb{R}$

Since (a_n) converges, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|a_n| < \frac{\varepsilon}{B} \quad \text{if } n > N$$

It follows that $|a_n b_n| < B \frac{\varepsilon}{B} = \varepsilon$ if $n > N$

This proves that $\lim_{n\to\infty} a_n b_n = 0$

(3) (a) FALSE $(a_n) = (n)$ is MONOTONE ($n+1 \ge n \ \forall n$), but divergent

(b) FALSE $(a_n) = (n)$ is MONOTONE. But we cannot extract any bounded sequence, since all INFINITE SEQUENCES DIVERGE

(c) FALSE Let $(a_n) = (-1)^n$, $(b_n) = (1 - \frac{1}{n})$.

Easier example: $(a_n) = (-1)^n$, $(b_n) = 1$

Note that $(-1)^n$ is bounded and $\lim b_n = 1$

However $(a_n b_n) = (-1)^n (1 - \frac{1}{n})$ does NOT CONVERGE.

Note that if $n = 2m$, then

$$(a_{2m} b_{2m}) = 1 - \frac{1}{2m} \to 1$$

if $n = 2m+1$, then

$$(a_{2m+1} b_{2m+1}) = (-1)\left(1 - \frac{1}{2m+1}\right) \to -1$$

Hence $(a_n b_n)$ does NOT CONVERGE

(d) FALSE. let $(a_n) = (n)$, $(b_n) = (\frac{1}{n})$.

then $(a_n b_n) = 1$. The sequence does NOT DIVERGE ~~TO ∞~~

(e) FALSE. SAME EXAMPLE as (d)

(f) TRUE
IF (a_n) is a Cauchy sequence, then it CONVERGES and it must be BOUNDED.

4 $S_1 = 2$, $S_2 = \frac{1}{4}(4-1) = \frac{3}{4}$, $S_3 = \frac{1}{4}(\frac{3}{2} - 1) = \frac{1}{8}, \ldots$

(a) $S_{n+1} = \frac{1}{4}(2S_n - 1)$

claim (S_n) is DECREASING $P_n: \boxed{S_{n+1} \leq S_n}$ P_n PROOF by INDUCTION

(1) $S_2 \leq S_1$

(2) Assume $S_{n+1} \leq S_n$ by step (2)

(3) $S_{n+2} = \frac{1}{4}(2S_{n+1} - 1) \leq \frac{1}{4}(2S_n - 1) = S_{n+1}$ ✓

claim (S_n) is bounded below by (-1) $P_n: \boxed{S_n \geq -1}$ PROOF by INDUCTION

(1) $S_1 \geq -1$

(2) Assume $S_n \geq -1$

(3) $S_{n+1} = \frac{1}{4}(2S_n - 1) \geq \frac{1}{4}(-3) = -\frac{3}{4} > -1$ ✓

SINCE (S_n) IS MONOTONE and BOUNDED, then it is CONVERGENT

(b) $L = \lim S_{n+1} = \lim \frac{1}{4}(2S_n - 1) = \frac{1}{4}(2L - 1)$

Hence $4L = 2L - 1 \Rightarrow \boxed{L = -\frac{1}{2}}$