Math 4331/6312 – Fall 2016 Name:

Test #2

This is a closed book test. Please, write clearly and justify all your steps, to get proper credit for your work. You can cite general theorems from the book if needed.

(1) [4 Pts] (a) State the definition of *completeness* in $\mathbb{R}^n$.
(b) Prove that a subset of $\mathbb{R}^n$ is complete if and only if it is closed.

(2) [4 Pts] Let $\{a\}$ be a singleton in $\mathbb{R}^2$ and $B \subset \mathbb{R}^2$ be closed. Suppose that a and B are disjoint. Let

$$d(a, B) = \inf\{\|a - b\| : b \in B\}.$$

Prove that $d(a, B) > 0$.

(3)[3 Pts] Prove that the set $S = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$ is closed.
[Hint: Use the definition of closed set to prove that any convergent sequence (x_n, y_n) in S will converge in S].

(4)[5 Pts] For each of the following statements, prove it (you can use the theorems discussed in class) or give a counterexample (in this case, you need to explain how your counterexample disproves the statement).

(a) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges.
(b) If $\sum_{n=1}^{\infty} a_n$ converges, $a_n \geq 0$ and b_n is bounded,, then $\sum_{n=1}^{\infty} a_n b_n$ converges.
(c) If $\{A_n\}$ is a countable collection of compact sets in $\mathbb{R}^2$, then $\cup_{n=1}^{\infty} A_n$ is compact.
(d) If $\{A_n\}$ is a countable collection of compact sets in $\mathbb{R}^2$, then $\cap_{n=1}^{\infty} A_n$ is compact.
(e) If A is a closed subset or $\mathbb{R}^2$ and B is an open subset of $\mathbb{R}^2$, then $A \setminus B$ is closed.

TEST #2 - SOLUTION

(1) (a) $S \subset \mathbb{R}^n$ is complete iff every Cauchy sequence ~~in~~ S converges in S

(b) ($\Rightarrow$) Suppose $S \subset \mathbb{R}^n$ is complete.

Let $x \in S$ be a limit point of S. Let $(x_n) \subset S$ be a sequence converging to S.

Since (x_n) is convergent it is also Cauchy. Hence it must converge in S.

This shows that any limit point of S must be in S, hence S is closed.

($\Leftarrow$) Suppose S is ~~complete~~ closed. Let (x_n) be a Cauchy seq. in S.

Since $S \subset \mathbb{R}^n$ and $\mathbb{R}^n$ is complete, then $(x_n) \to x \in \mathbb{R}^n$. However x is also a limit point of S, since $(x_n) \subset S$. Since S is closed, then x must belong to S. This implies that S is complete.

(2) Since a is a singleton, and a is disjoint from B, then there exists a ball $B_\delta(a)$ s.t. $B_\delta(a) \cap B = \emptyset$. ~~Hence ... $\geq \delta \ \forall b \in B$.~~ If NOT, there would be a sequence $(b_n) \subset B$ converging to a and, since B is closed, a would not be disjoint from B. ~~This implies that $d(A,B) \geq \delta > 0$.~~ This implies that $d(a,B) \geq \delta > 0$

(3) Let (x_n, y_n) be a convergent sequence in S, say $\lim(x_n, y_n) = (x, y) \in \mathbb{R}^2$

$\lim x_n = x$, $\lim y_n = y$.

It follows from the definition that if $x_n \geq 0$, then $x \geq 0$, and if $y_n \geq 0$, then $y \geq 0$. Similarly if $x_n < 0, y_n < 0$

This implies that $\lim x_n y_n = xy \geq 0$. Hence S contains all its limit points and it must be closed

(4) (a) F: Let $a_n = \frac{1}{n^2}$. $\sum a_n$ converges, $\sum \sqrt{a_n}$ does NOT

(b) T: $|b_n| \leq M$ for some $M > 0$. Hence $|a_n b_n| \leq M a_n$ and $\sum a_n b_n \leq M \sum a_n < \infty$

(c) F: Let $A_n = [-n, n]^2$. $\cup A_n$ is unbounded, hence cannot be compact

(d) T: Since A_n are compact set, they are closed and bounded. Hence $\cap A_n$ is also closed and bounded. Hence $\cap A_n$ is compact

(e) T: $A \setminus B = A \cap B^c$. A is closed, B^c is also closed, as it is the complement of an open set. Hence $A \setminus B$, being the INTERSECTION of 2 closed sets must be ~~compact~~ closed.

---

(2) [Alternate] Suppose by contradiction that $d(A,B) = 0$. Then we can find a sequence $(b_n) \subset B$ s.t. $\lim b_n = a$. However B is a closed set containing all its limit points, and it cannot have ~~as~~ limit point ~~a~~ a point disjoint from itself. Hence it must be $d(A,B) > 0$.