Math 4331/6312 – Fall 2016 Name:

Test #3

This is a closed book test. Please, write clearly and justify all your steps, to get proper credit for your work. You can cite general theorems from the book if needed.

(1) [3 Pts] Prove that if $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuous and there are $x \in \mathbb{R}^n$ and $M \in \mathbb{R}$ such that $|f(x)| < M$, then there is an $r > 0$ such that $|f(y)| < M$ for all $y \in B_r(x)$.

(2) [4 Pts] Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

(a) Prove that f attains a maximum or a minimum on $\mathbb{R}$.

(b) Does f need to attain both a maximum and a minimum on $\mathbb{R}$? Prove it or give an example of such a function attaining only a mimimum or only a maximum on $\mathbb{R}$.

(3)[3 Pts] Let $f : [0, 1] \mapsto \mathbb{R}$ be a continuous and one-to-one function such that $f(0) < f(1)$. Prove that f is strictly increasing in $[0, 1]$.

(4)[5 Pts] For each of the following statements, prove it (you can use the theorems discussed in class) or give a counterexample (in this case, you need to explain how your counterexample disproves the statement).

(a) If f and g are increasing functions on an interval $I \subset \mathbb{R}$, then $f + g$ is increasing on I.

(b) If f and g are increasing functions on an interval $I \subset \mathbb{R}$, then $f\,g$ is increasing on I.

(c) If f is one-to-one then it is continuous.

(d) If f is continuous and bounded then it is Lipschitz.

(e) If f is Lipschitz then it is bounded.

TEST #3 SOLUTION

(1) By continuity, given any $\varepsilon>0$, $\exists\ r>0$ s.t. $|f(x)-f(y)|<\varepsilon$ if $\|x-y\|<r$.

Since $|f(x)|\cancel{<}M$, there is a $\delta>0$ s.t. $|f(x)|\leq M-\delta$.

Choose $\varepsilon<\delta$. This implies that $|f(y)|\leq|f(y)-f(x)|+|f(x)|<M-\delta+\varepsilon<M$

for all $y: \|x-y\|<r$.

(2) (a) Since $\lim\limits_{x\to\pm\infty} f(x)=0$, ~~there is an $R\in\mathbb{R}$~~ for any $\varepsilon>0$, there is an $R\in\mathbb{R}$ s.t.

$|f(x)|<\varepsilon$ if $|x|>R$.

For $x\in[-R,R]$, $f(x)$ attains it maximum and minimum in $[-R,R]$, say that there are $a,b\in[-R,R]$ s.t. $M=\max\limits_{x\in[-R,R]} f(x)=f(a)$, $m=\min\limits_{x\in[-R,R]} f(x)=f(b)$. If $f\equiv 0$, proof is complete.

Assume f NOT IDENTICALLY 0.

If $M>0$, then $x=a$ is also the maximum location for $f(x), x\in\mathbb{R}$; if $m<0$, then $x=b$ is also the minimum location for $f(x)$, $x\in\mathbb{R}$.

$M\geq 0$ and $m\leq 0$ IS NOT POSSIBLE, otherwise f would be identically 0. $M\leq 0$ and $m\leq 0$ or $M\leq 0$ and $m\geq 0$ is not possible, otherwise f would be identically 0.

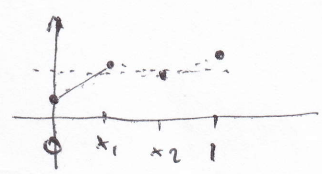
This shows that there must be a maximum value or a minimum value on $\mathbb{R}$.

(b) $f(x)=e^{-x^2}$. It achieves a max in $\mathbb{R}$, not a minimum.

(3) Arguing by contradiction, suppose there are $x_1,x_2\in(0,1)$ with $x_1<x_2$ s.t. $f(x_1)\geq f(x_2)$.

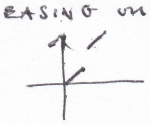
Clearly $f(x_1)=f(x_2)$ is NOT POSSIBLE otherwise f would not be one-to-one.

Examine now the situation when $f(x_1)>f(x_2)$. By IVT $f(x)$ attains all values between $[f(0),f(x_1)]$ in $(0,x_1)$ and, all values in $[f(x_2),f(x_1)]$ in (x_1,x_2) and all values in $[f(x_2),f(1)]$ in $(x_2,1)$. Since $f(0)<f(1)$ and $f(x_2)<f(x_1)$, it follows that the intervals $[f(0),f(x_1)]$, $[f(x_2),f(x_1)]$, $(f(x_2),f(0)]$ must overlap. This violates the assumption that f is one-to-one. Hence it must be $f(x_1)<f(x_2)$. That is, f must be strictly INCREAS.



(4) (a) TRUE $\quad x_1\leq x_2$, $f(x_1)\leq f(x_2)$, $g(x_1)\leq g(x_2) \Rightarrow f(x_1)+g(x_1)\leq f(x_2)+g(x_2)$

(b) FALSE $\quad f(x)=x$, $g(x)=x-1$. $\quad f(x)g(x)=x(x-1)$ NOT INCREASING on $(-2,2)$



(c) FALSE $\quad f(x)=x$, $x\in[0,1)$, $\quad f(x)=x+2$, $x\in[1,2]$

(d) FALSE $\quad f(x)=\sqrt{x}$, $x\in[0,1]$. Clearly $f(x)$ is bounded.

$|f(x)-f(0)|=\sqrt{x}\leq M|x-0|=Mx \iff x^{-1/2}\leq M$ $\quad$ This IS NOT POSSIBLE

Since $\lim\limits_{x\to 0^+} x^{-\frac{1}{2}}=\infty$. This shows that there is NO constant M such that

$|f(0)-f(x)|\leq M|0-x|$.

(e) FALSE $\quad f(x)=2x$ is Lipschitz in $\mathbb{R}$, but it is unbounded in $\mathbb{R}$.