HW1 SOLUTION

7.1 C Let $f \in C'[a,b]$

- $\rho(f) = \|f'\|_\infty = \sup\{|f'(x)| : x \in [a,b]\} \geq 0$
- $\rho(\alpha f) = \|\alpha f'\|_\infty = \sup\{|\alpha f'(x)| : x \in [a,b]\} = |\alpha| \sup\{|f'(x)| : x \in [a,b]\} = |\alpha| \|f\|_\infty$
- $\rho(f+g) = \|f'+g'\|_\infty = \sup\{|f'(x)+g'(x)| : x \in [a,b]\} \leq \sup\{|f'(x)| + |g'(x)| : x \in [a,b]\}$

$\leq \sup\{|f'(x)| : x \in [a,b]\} + \sup\{|g'(x)| : x \in [a,b]\}$

$= \rho(f) + \rho(g)$

- $\rho(f)$ is NOT a norm, since it does NOT satisfy positive-definiteness.

Let $f(x) = 1$. Then $\rho(f) = 0$.

7.1 D For $x, y \in V$, $\|x\| \leq \|x-y\| + \|y\| \Rightarrow \|x\| - \|y\| \leq \|x-y\|$

$\|y\| \leq \|x-y\| + \|x\| \Rightarrow \|y\| - \|x\| \leq \|x-y\|$

Thus $|\|x\| - \|y\|| \leq \|x-y\|$

7.1 E Let x, y be in the unit ball of $(V, \|\cdot\|)$.

Let $z = tx + (1-t)y$ where $t \in [0,1]$.

$\|z\| = \|tx + (1-t)y\| \leq t\|x\| + (1-t)\|y\| \leq t + 1 - t = 1$

This shows that the line segment connecting x to y is contained in the unit ball.

7.2 B Let (x_n) be a convergent sequence in a NORMED space $(V, \|\cdot\|)$

Let $\lim x_n = x \in V$

Hence, given $\varepsilon > 0$, $\exists N$ s.t. $\|x - x_n\| < \varepsilon/2$ if $n > N$

Hence, $\|x_n - x_m\| \leq \|x_n - x\| + \|x - x_m\| < \varepsilon$ if $n, m > N$. Hence (x_n) is Cauchy

7.2 D Let $A \subset V$ and $U \subset V$ be open

Since U is open, given any $u \in U$, $\exists r > 0$ s.t. $B_r(u) \subset U$ $\{v \in V : \|v - u\| < r\}$

Given any $a \in A$, $a + u \in a + U = \{a + u : u \in U\}$

claim: $B_r(a+u) \subset a + U$

In fact $v \in V$ satisfies $v \in a + U \Leftrightarrow (v - a) \in U$

$B_r(a+u) = \{v \in V : \|v - a - u\| < r\}$

$v \in B_r(a+u) \Leftrightarrow v' := (v-a) \in \{v' \in V : \|v' - u\| < r\} = B_r(u)$

Since $B_r(a+u) \subset a + U$ for any $a \in A$, then the set $A + U$ is open, $A + U = \bigcup_{a \in A} (a + U)$ ~~countable~~ union of open sets

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 # REMARK $v \in a + U \Leftrightarrow v - a \in U$

For every a, r, $B_r(a+u) = a + B_r(u)$

If $v \in a + U$, then $v = a + u$ for some $u \in U$

Since U open, $\exists r > 0$: $B_r(u) \subset U$. Hence $B_r(a+u) = a + B_r(u) \subset a + U$. Hence $a + U$ is open

7.2.6 Let $\mathbb{K} \subset V$ ~~be~~ compact, where $(V, \|\cdot\|)$ is a NORMED SPACE.

This implies that, given any sequence $(x_n) \subset \mathbb{K}$, there is a subsequence $(x_{n_k}) \to x \in \mathbb{K}$.

- Suppose c is a limit point of $\mathbb{K}$, that is $c = \lim(x_n)$ where $(x_n) \subset \mathbb{K}$.
  Then $\exists$ subsequence $(x_{n_k}) \subset \mathbb{K}$ with $\lim x_{n_k} = c \in \mathbb{K}$. Hence $\mathbb{K}$ is closed.
- Arguing by contradiction, suppose $\mathbb{K}$ is unbounded.
  Then $\exists$ $(x_n) \subset \mathbb{K}$ s.t. $\|x_n\| > n$ for all $n \geq 1$.
  If (x_n) has a convergent subsequence $(x_{n_k}) \subset \mathbb{K}$ with limit c in $\mathbb{K}$, then

  $$\|c\| = \lim \|x_{n_k}\| \geq \lim n_k = \infty. \Rightarrow \text{CONTRADICTION}.$$

  Hence $\mathbb{K}$ must be bounded.

7.2.14 (a) Let $\mathbb{K} \subset V$ be compact, where $(V, \|\cdot\|)$ is a NORMED SPACE

~~By 7.2.6 $\mathbb{K}$ is also closed.~~

Given any Cauchy sequence in $\mathbb{K}$, say (x_n), this sequence has a convergent subsequence (x_{n_k}) with $\lim x_{n_k} = \bar{x} \in \mathbb{K}$, since $\mathbb{K}$ is compact.

~~Given any $\varepsilon > 0$~~ Since, $\|x_n - \bar{x}\| \leq \|x_n - x_{n_k}\| + \|x_{n_k} - \bar{x}\|$, it follows that $\lim x_n = \bar{x}$, hence $\mathbb{K}$ is complete.

(b) Let $B \subset V$ be closed, where $(V, \|\cdot\|)$ is a COMPLETE NORMED SPACE.

Given any Cauchy sequence $(x_n) \subset B$, we have that $\lim x_n = x \in V$, since V is complete.

However x is a limit point of B and B is closed. Hence it must be $x \in B$.