HW #2 SOLUTION

7.3 A Let V be n-dim and $\{e_1, \dots, e_n\}$ be a l.i. set.

Hence, for any $v \in V$, $v = \sum \alpha_i e_i$ for some $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$

By Prop 7.3.1, $\exists\ 0 < c_1 \le c_2 < \infty$ and $0 < c_3 \le c_4 < \infty$ s.t.

$$c_1 \|\alpha\|_2 \le \|v\| \le c_2 \|\alpha\|_2 \quad \text{and} \quad c_3 \|\alpha\|_2 \le |||v||| \le c_4 \|\alpha\|_2$$

Hence $|||v||| \le \frac{c_4}{c_1} \|v\|$ and $|||v||| \ge \frac{c_3}{c_2} \|v\|$.

7.3 D Let $(w_i) \subset W$ be Cauchy. Let $T : W \to V$ be a Lipschitz map.

Hence, for any $w_1, w_2 \in W$, $\|Tw_1 - Tw_2\| \le C \|w_1 - w_2\|$

It follows that if $\|w_k - w_n\| < \varepsilon$ then $\|Tw_k - Tw_n\| < C\varepsilon$, hence

the seq. (Tw_k) is Cauchy.

7.4.B $\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$

$\|x-y\|^2 = \langle x-y, x-y \rangle = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$

Hence $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

7.4 I $Tx = 0 \Rightarrow x = 0$ iff T is INVERTIBLE

Hence $\langle x, x\rangle_T = \langle Tx, Tx\rangle = \|Tx\|^2 = 0$ implies $x = 0$ iff T INVERTIBLE

This shows that $\langle x, y\rangle_T$ is NOT an INNER PRODUCT if T is NOT INVERTIBLE

If T is INVERTIBLE, positivity holds

Symmetry $\langle x, y\rangle_T = \langle Tx, Ty\rangle = \langle Ty, Tx\rangle = \langle y, x\rangle_T$

bilinearity $\langle \alpha x + \beta y, z\rangle = \langle \alpha Tx + \beta Ty, Tz\rangle = \alpha\langle Tx, Tz\rangle + \beta\langle Ty, Tz\rangle =$

$= \alpha\langle x, z\rangle_T + \beta\langle y, z\rangle_T$

7.4 J $\langle A, A\rangle = \mathrm{Tr}(AA^t) = \sum_i \sum_j a_{ij}^2 \ge 0$

$= 0$ iff all $a_{ij} = 0$

$\langle A, B\rangle = \mathrm{Tr}(A B^t) = \mathrm{Tr}(BA^t) = \langle B, A\rangle$

$\langle A + C, B\rangle = \mathrm{Tr}((A+C)B^t) = \mathrm{Tr}(AB^t + CB^t) = \mathrm{Tr}(AB^t) + \mathrm{Tr}(CB^t) = \langle A, B\rangle + \langle C, B\rangle$

7.5 A (a)
- $\mathrm{span}(f_1) = \mathrm{span}(x_1)$ since $f_1 = x_1/\|x_1\|$
- suppose $\mathrm{span}\{f_1, \dots, f_k\} = \mathrm{span}\{x_1, \dots, x_k\}$
- $\mathrm{span}\{f_1, \dots, f_{k+1}\} = \mathrm{span}\{f_1, \dots, f_k, x_{k+1}\}$ since $x_{k+1} \in \mathrm{span}\{x_{k+1}, f_1, \dots, f_k\}$

$= \mathrm{span}\{x_1, \dots, x_k, x_{k+1}\}$

(b) ~~$\langle y_i, y_j\rangle = \langle x_i - \sum_{\ell=1}^{i-1} \langle x_i, f_\ell\rangle f_\ell,\ x_j - \sum_{n=1}^{j-1} \langle x_j, f_n\rangle f_n\rangle = \langle x_i, x_j\rangle +$~~

~~Assume $i \ge j$ $\quad - \langle x_i, \sum_{n=1}^{j-1} \langle x_j, f_n\rangle f_n\rangle - \langle x_j, \sum_{\ell=1}^{i-1} \langle x_i, f_\ell\rangle f_\ell\rangle + \sum\sum \langle x_i, f_\ell\rangle \langle x_j, f_n\rangle \langle f_n, f_\ell\rangle$~~

~~$i \ge j$~~

(b) By induction $y_2 = x_2 - \langle x_2, f_1\rangle f_1$ Hence $y_2 \perp f_1$

$f_2 \perp f_1$

Suppose $y_k \perp \{f_1, \dots f_{k-1}\}$ span$\{y_k\}$ = span$\{f_k\}$

then $y_{k+1} = x_{k+1} - \sum_{i=1}^{k} \langle x_{k+1}, f_i\rangle f_i \perp$ span$\{f_1, \dots f_k\}$ =

= span $\{f_1, \dots f_{k-1}\} \cup \{f_k\}$

$\perp \{f_1, \dots f_{k-1}\} \cup \{f_k\}$

This shows that All $\{f_i\}$ are orthogonal.

NORMALIZATION holds by construction

(c) (a)+(b) imply that $\{f_1, \dots f_n\}$ is ON set which spans $\{x_1, \dots, x_n\}$

Hence it is an ONB