HW 3 SOLUTION

7.6 B

$$\int_0^1 \sin n\pi x \sin m\pi x \, dx = \frac{1}{2}\int_0^1 \left[\cos(n-m)\pi x - \cos(n+m)\pi x\right] dx$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ 1/2 & \text{if } n = m \end{cases}$$

$\{\sqrt{2}\sin n\pi x : n \in \mathbb{N}\}$ is an ON set

7.6 F(a)

$$f(\theta) = a_0 + \sum a_k \cos k\theta + b_k \sin k\theta$$

$$f(\theta+\alpha) = A_0 + \sum A_k \cos k\theta + B_k \sin k\theta$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta+\alpha)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta = a_0 \qquad \theta - \alpha = u$$

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta+\alpha)\cos k\theta\, d\theta = \frac{1}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(u)\cos k(u-\alpha)\, du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)(\cos ku \cos k\alpha + \sin ku \sin k\alpha)\, du$$

$$= \cos k\alpha \, a_k + \sin k\alpha \, b_k$$

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta+\alpha)\sin k\theta\, d\theta = \frac{1}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(u)\sin k(u-\alpha)\, du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)(\sin ku \cos k\alpha\, du - \cos ku \sin k\alpha)\, du$$

$$= \cos k\alpha \, b_k - \sin k\alpha \, a_k$$

$$f(\theta+\alpha) = a_0 + \sum a_k \cos k\alpha \cos k\theta + b_k \sin k\alpha \cos k\theta + b_k \cos k\alpha \sin k\theta + {}$$
$$- a_k \sin k\alpha \sin k\theta$$

$$\boxed{= a_0 + \sum a_k \sin k(\theta-\alpha) + b_k \cos k(\theta-\alpha)}$$

7.6 G

$$f(\theta) = a_0 + \sum a_k \cos k\theta + b_k \sin k\theta$$

(a) $g(\theta) = f(-\theta) = A_0 + \sum A_k \cos k\theta + B_k \sin k\theta$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta = a_0$$

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(-\theta)\cos k\theta\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos ku\, du = a_k$$

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(-\theta)\sin k\theta\, d\theta = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\sin ku\, du = -b_k$$

$$\boxed{g(\theta) = a_0 + \sum a_k \cos k\theta - b_k \sin k\theta}$$

(b) Assume $h(\theta) = h(\pi - \theta)$ $\qquad h(\theta) = A_0 + \sum_k A_k \cos k\theta + B_k \sin k\theta$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\pi-\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} h(u)\, du = a_0$$

$\pi - \theta = u \quad \Rightarrow \theta = \pi - u$

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos k\theta\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\pi-\theta)\cos k\theta\, d\theta = \frac{1}{\pi}\int_0^{2\pi} h(u)\cos k(\pi-u)\, du$$

$$= \frac{1}{\pi}\int_0^{2\pi} h(u)\cos(ku - k\pi)\, du = \begin{cases} A_k & \text{if } k \text{ even} \\ -A_k & \text{if } k \text{ odd}\end{cases}$$

~~SAME FOR~~ B_k

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin k\theta\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\pi-\theta)\sin k\theta\, d\theta = \frac{-1}{\pi}\int_0^{2\pi} h(u)\sin(ku - k\pi)\, du$$

$$= \begin{cases} -B_k & \text{if } k \text{ even} \\ B_k & \text{if } k \text{ odd}\end{cases}$$

Hence it must be $A_k = 0$ if k odd, $B_k = 0$ if k even

7.6 I $\quad$ If f is odd, then $f(x)\cos kx$ is odd and the Lens

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx = 0, \qquad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

If f is even, then $f(x)\sin kx$ is oded. Hence

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx = 0$$

7.7 C $\quad$ (a) Let $M \subset H$ be closed and denote by P the orthogonal projection of H onto M. For any $x \in H$, we can write:

$$x = Px + (I-P)x.$$

We have that $Px \in M$ and $(I-P)x \in M^\perp$. In fact $P((I-P)x) = 0$ showing that $(I-P)x \in N(P)$ so that $(I-P)x \in M^\perp$

Since Px and $(I-P)x$ are orthogonal, then

$$\|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$$

To show uniqueness, let $x = m' + y'$, where $m' \in M$, $y' \in M^\perp$

Then $Px + (I-P)x = m' + y'$

Then $Px - m' = y' - (I-P)x$ $\quad$ Since $M \cap M^\perp = \{0\}$, it must be

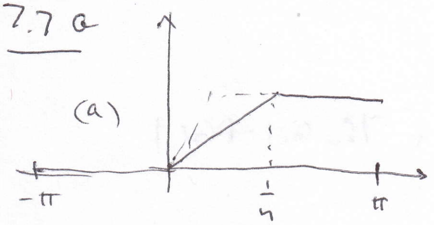
$Px = m'$, $\quad (I-P)x = y'$.

(b) If $m \in M$, $y \in M^\perp$, then $\langle m, y\rangle = 0$, hence $m \in (M^\perp)^\perp$ and $M \subset (M^\perp)^\perp$

If $m \in (M^\perp)^\perp$, $y \in M^\perp$, then $\langle m, y\rangle = 0$ and $m \in M$. Hence $(M^\perp)^\perp \subset M$.

7.7 C

(a)



$$\| f_n(x) - 1 \|^2 = \int_0^{1/n} (1-nx)^2\, dx =$$

$$= \cancel{\ldots} \frac{(1-nx)^3}{3n}\Big|_0^{1/n} = \frac{1}{3n} \to 0 \quad \text{as } n \to \infty$$

This shows that $f_n \to \chi_{[0,\pi]}$

(b) If $h \in C[-\pi,\pi]$, then h cannot contain a jump discontinuity at $x=0$

If ~~...~~ $h(0)=c>0$ then, ~~...~~ $\varepsilon>0$, $|h(x) - h(x+\Delta)| < \frac{\varepsilon}{2}$ if $|\Delta|<r$.

Hence $|h(x+\Delta)| > \frac{c}{2}$ for $|\Delta|<r$.

Hence $\int |\chi_{[0,\pi]} - h|^2 \geq \int_{-r}^{0} \frac{c^2}{2} + \int_0^r \left(1-\frac{c}{2}\right)^2 > 0$

Also if $h(0)=0$, then $|h(x+\Delta)| < \varepsilon$ for $|\Delta|<r$

In this case $\int |\chi_{[0,\pi]} - h|^2 \geq \int_0^r |1-\varepsilon|^2 > 0$

(c) ~~By part c, there is no sequence of continuous functions converging to~~ $\chi_{[0,\pi]}$

~~in the uniform norm. In particular,~~

Since $(f_n) \to \chi_{[0,\pi]}$, there is a Cauchy seq. (f_n) whose limit is not in $C[-\pi,\pi]$. Thus $C[-\pi,\pi]$ is NOT COMPLETE in the L^2-norm

8.1 A

$$\lim_{n\to\infty} x n e^{-nx} = 0 \quad \text{for all } x \geq 0$$

This is true since $\lim_{z\to\infty} z e^{-z} = 0$

For any n,
$$\int_0^{1/n} f_n(x)\,dx = \int_0^{1/n} x n e^{-nx}\, dx = \left[-x e^{-nx} - \frac{1}{n} e^{-nx}\right]_0^{1/n} =$$

$$= -\frac{1}{n} e^{-1} - \frac{1}{n} e^{-1} + \frac{1}{n} = \frac{1}{n}\left(-2e^{-1}+1\right) = \frac{1}{n}\,\frac{e-2}{e}$$

This implies that there's $y \in [0,\frac{1}{n}]$ s.t. $f_n(y) > \frac{e-2}{e}$ then

This CONTRADICTS UNIFORM CONVERGENCE

[Also one can observe that $f_n(\frac{1}{n}) = \frac{1}{e} > 0$ $\forall n$.

8.1 H

By ass. $|f_k(x) - f_k(y)| \leq M|x-y|$ thus any f_k is Lipschitz continuous, hence UNIF. CONTINUOUS

Since $f_k(t) \to f(t)$ pointwise, thus also f is Lipschitz contin and UNIF. CONTINUOUS

Pick points $x_1, x_2 \ldots x_m$ in $[0,1]$ whose distance is $\frac{\varepsilon}{3M}$ from each other

For each $i=1,\ldots,m$ find N_i s.t. $|f_n(x_i) - f(x_i)| < \varepsilon/3$

Set $N = \max N_i$

Pick i s.t. $|x - x_i| < \frac{\varepsilon}{3M}$.

then $|f(x) - f_n(\cancel{x})| \le |f(x) - f(x_i)| + |f(x_i) - f_n(x_i)| + |f_n(x_i) - f(x)|$

By Lipschitz property $|f(x) - f(x_i)| \le M|x - x_i| < \frac{\varepsilon}{3}$

$|f_n(x_i) - f_n(x)| \le M|x - x_i| < \frac{\varepsilon}{3}$

Also $|f(x_i) - f_n(x_i)| < \frac{\varepsilon}{3}$

This shows that $|f(x) - f_n(x)| < \varepsilon$ INDEP of x provided $n > N$