HW # 4 SOLUTION

8.2 D Since $(f_n) \to f$ UNIFORMLY, $\|f\|_\infty < \infty$, say $\|f\|_\infty = M_1$

Since $(g_n) \to g$ uniformly, (g_n) is also bnded, say $\|g_n\|_\infty < M_2$ $\forall n$.

Given $\varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $\|f_n - f\|_\infty < \varepsilon/2M_2$ if $n > N_1$ and

$\exists N_2 \in \mathbb{N}$ s.t. $\|g_n - g\|_\infty < \varepsilon/2M_1$ if $n > N_2$.

Let $N = \max(N_1, N_2)$. Then

$$\|f_n g_n - fg\|_\infty \le \|f_n g_n - f g_n + f g_n - f g\|_\infty \le \|f_n - f\|_\infty \|g_n\|_\infty + \|f\|_\infty \|g - g_n\|_\infty$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

8.2 G Let $\|f_k\|_\infty < A_k$

Since $(f_n) \to f$ uniformly, $\exists N \in \mathbb{N}$ s.t. $\|f_n - f\|_\infty < 1$ if $n > N$

Thus $\|f_n\|_\infty < \|f\| + 1$ if $n > N$

Therefore $\|f_k\|_\infty < \max\{A_1, \ldots, A_N, \|f\| + 1\} < \infty$.

8.3 C Since $g \in C[0,1]$, $\exists M > 0$ s.t. $\|g\|_\infty < M$.

Since $(f_n) \to f$ uniformly, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\|f_n - f\|_\infty < \varepsilon/M$ if $n > N$

Thus $$\left| \int_0^1 f_n(t)\, g(t)\, dt - \int_0^1 f(t)\, g(t)\, dt \right|$$
$$\le \int_0^1 |f_n(t) - f(t)|\,|g(t)|\, dt \le M \int_0^1 |f_n(t) - f(t)|\, dt < M \frac{\varepsilon}{M} < \varepsilon$$

8.4 E By Weierstrass M-test then, since $|a_n \cos nx| \le |a_n|$ and $\sum |a_n| < \infty$,

it follows that $\sum a_n \cos nx$ converges UNIFORMLY on $\mathbb{R}$.

8.5 C By prior result, if $\lim \frac{a_{n+1}}{a_n} = L$, then $\lim (a_n)^{1/n} = L$

Using Hadamard's theorem, it follows that $\sum a_n x^n$ has RADIUS of convergence L

8.6 A $f_n(x) = x^n$, $x \in [0,1]$ is a sequence in $C[0,1]$.

We find that $\lim_{n \to \infty} x^n = \chi_{\{1\}}(x)$.

This sequence has no uniform convergent subsequence. In fact, if there were a unif. convergent subsequence, it would converge to $\chi_{\{1\}}(x)$, but this function is NOT CONTINUOUS, hence CANNOT be a UNIFORM LIMIT.

This shows that there is a sequence in $C[0,1]$ with no convergent subsequence. This IMPLIES that $C[0,1]$ is NOT COMPACT.