HW #5 SOLUTION

10.2 A

Let $f \in C[a,b]$.

For $t \in [0,1]$, define $g(t) = f(a+(b-a)t)$

By Weierstrass thm., there exists a sequence (p_n) of polynomials on $[0,1]$

s.t. $\lim |p_n(t) - g(t)| = 0$ UNIFORMLY in $[0,1]$

Let $x = a+(b-a)t$. Then $t = \frac{x-a}{b-a}$

Then $\lim | p_n(\frac{x-a}{b-a}) - g(\frac{x-a}{b-a})| = \lim | p_n(\frac{x-a}{b-a}) - f(x) | = 0$ UNIF. for $x \in [a,b]$

This shows that the seq. of polynomials $(p_n(\frac{x-a}{b-a}))$ converge UNIF. to $f \in C[a,b]$

10.2 D

$f^2(x) \in C[0,1]$. $f \in C[0,1]$ and it is bounded on $[0,1]$

By Weierstrass thm., there is a seq. of polynomials $p_n(x) = \sum_{k=0}^{m_n} a_k^{(n)} x^k$

converging unif. to f.

$$\int_0^1 f^2(x)\,dx = \int_0^1 f(x) \lim p_n(x)\,dx = \lim_n \sum_{k=0}^{m_n} a_n^{(n)} \int_0^1 f(x) x^k\,dx$$

We can INTERCHANGE LIMIT and INTEGRAL since seq. converges UNIFORMLY and f is bounded.

Since $\int_0^1 f(x) x^k = 0$ $\forall k$, then $\int_0^1 f^2(x)dx = 0$ and $f = 0$.

10.2 F

Since $f' \in C[0,1]$, we can use Weierstrass thm.

There exists a seq. (q_n) of polynomials s.t. $(q_n) \to f'$ UNIFORMLY.

We have: $f(x) = \int_0^x f'(u)\,du + f(0)$

then $f(x) = \lim_{n\to\infty} \int_0^x q_n(u)\,du + f(0)$ UNIFORMLY

Let $p_n(x) = \int_0^x q_n(u)\,du + f(0)$. Note: $p_n'(x) = q_n(x)$

This shows that there is seq. $(p_n) \to f$ UNIFORMLY.

10.8 C Let $f \in S_1(\Delta)$

For any subinterval $[x_i, x_{i+1}]$, $f|_{x_i, x_{i+1}} = m_i x + p_i$

By the continuity of linear splines, for each $x_i \in \Delta$,

$$m_i x_i + p_i = m_{i+1} x_i + p_{i+1} \qquad 1 \le i \le k-1 \qquad (1)$$

Since f' is continuous, $f'|_{x_i, x_{i+1}} = f'|_{x_{i+1}, x_{i+2}}$, so that

$$m = m_i \quad \forall i$$

By eq. (1), it must also be $p_i = p_{i+1} \ \forall i$

Thus $f = mx + p$

10.8 D (a) By induction, we can show that (p_k) defined by $p_0 = 0$, $p_{k+1} = p_k + \frac{x - p_k^2}{2}$ converges to $\sqrt{x}$. It follows that the seq. $(q_k) = (p_k^2)$ converges uniformly to $|x|$ on $[a,b]$

(b) By exercise 10.2 A, $\mathrm{abs}_a(x) = |x-a|$ is a unif. limit of polynomials

(SKETCHY)

(c) $\mathrm{Span}\{1, \mathrm{abs}_a : a \in \Delta\} = \{$piecewise linear functions of form $f(x) = m_0 + \sum_{x_i \in \Delta} m_i |x - x_i|\}$

Let f as above.

Since f is a sum of continuous functions, it is continuous.

Since f is a sum of linear functions, it is piecewise linear.

The derivative of f ~~only~~ can only change at points $x = x_i \in \Delta$.

Hence any $f \in S_1(\Delta)$ is contained in $\mathrm{span}\{1, \mathrm{abs}_a : a \in \Delta\}$.

(d) Let $f \in C[a,b]$. Since f is continuous, given $\varepsilon > 0$, we can find $r > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ provided $|x - y| < r$.

~~$\sup |f(x) - f(y)| < \varepsilon$~~ $\omega(f, r) = \sup\{|f(x) - f(y)| : |x - y| < r\} < \varepsilon$

Then we can find a line $\ell(x)$ s.t. $|f(x) - \ell(x)| < \varepsilon$ if $x \in I_x$, where I_x is an interval of size $|I_x| < r$

By lemma 10.8.1, the linear spline approximation $J_1 f$ satisfying

$$\|f - J_1 f\|_\infty \le 2\varepsilon$$