HW 6 SOLUTION

11.1 B $Tx = \sin x$, $x \in [-1,1]$. T is continuous.

By MVT, $|Tx - Ty| = |T'(z)||x-y|$ where $z \in [-1,1]$

and $|T'(z)| = \dfrac{|Tx-Ty|}{|x-y|}$

Suppose T is a contraction, hence $\exists\, c<1$ s.t. $|Tx-Ty| < c|x-y|$ $\forall x,y \in [-1,1]$

However, $\lim\limits_{x \to y} \dfrac{|Tx-Ty|}{|x-y|} = \lim\limits_{z \to 0} |T'(z)| = 1$ since $T'(z) = \cos z$

Hence for $|x-y|$ sufficiently small, we can make $\dfrac{|Tx-Ty|}{|x-y|}$ arbitrarily close to 1, contradicting the assumption that T is a contraction.

11.1 E $\|STx - STy\| = \|S(Tx) - S(Ty)\| \le s\|Tx - Ty\| \le st\|x-y\|$

This shows that ST is a contraction with Lipschitz constant st.

Note that $st < 1$ since $s<1$ and $t<1$

11.1 I By triangle inequality,

$$\|x_s - x_t\| = \|x_s - Sx_t + Sx_t - x_t\| \le \|x_s - Sx_t\| + \|Sx_t - x_t\|$$

$$= \|Sx_s - Sx_t\| + \|Sx_t - Tx_t\|$$

$$\le c\|x_s - x_t\| + \sup_{x \in X} \|Sx - Tx\|$$

$$= c\|x_s - x_t\| + \|S-T\|_\infty$$

Thus $\|x_s - x_t\| \le (1-c)^{-1}\|S-T\|_\infty$

11.2 A Let $Tx = x - \dfrac{f(x)}{f'(x_0)}$, $f \in C^2$

Suppose $\exists\, x^* \in \mathbb{R}$ s.t. $f(x^*) = 0$, $f'(x^*) \neq 0$

Given $\varepsilon > 0$, $\exists\, r > 0$ s.t. $|f'(x^*) - f'(y)| < \varepsilon$ for $y \in (x^*-r, x^*+r)$

Hence $|T'(x^*)| = \left|1 - \dfrac{f'(x^*)}{f'(x_0)}\right| = \left|\dfrac{f'(x_0) - f'(x^*)}{f'(x_0)}\right| < \dfrac{\varepsilon}{|f'(x_0)|}$ if $x_0 \in (x^*-r, x^*+r)$

Since ε is arbitrary and $|f'(x_0)| > 0$, we can choose ε such that $|T'(x^*)| < 1$

Thus, by Lemma 11.1.2 there is an open set $U = (x^*-r, x^*+r)$ such that for every $x_0 \in U$ the sequence $x_n = T^n x_0$ converges to x^*

11.3 A Let x^* be a point of period n, that is $T^n x^* = x^*$.

For any $i = 0, 1, \ldots, n-1$, $T^{n+i} x^* = T^i T^n x^* = T^i x^*$

This shows that also $T^i x^*$ is a periodic point.

If x^* is an attractive point, there exists an open set $U \subset X$ containing x^* such that for every $x \in U$ the sequence $(T^{kn} x)$ converge to x^*

Now for any $x \in U$, $\lim_{k\to\infty} (T^{kn} T^i x) = T^i \lim_{k\to\infty} (T^{kn} x) = T^i x^*$

This shows that $T^i x^*$ is also attracting

If x^* is repelling, $\exists\, U \subset X$, with $x^* \notin U$ s.t. the orbit $O(x^*)$ leaves U

$\lim_{k\to\infty} T^{kn} T^i x = T^i \lim_{k\to\infty} T^{kn} x$. If $x = T^{kn} x$ leaves eventually the set U,

so $T^i T^{kn} x$ will eventually leave the set TU.

Thus $T^i x^*$ is also repelling