Math 4332/6313 – Spring 2017 Name:

<u>Test #2</u>

This is a closed book test. Please, write clearly and justify all your steps, to get proper credit for your work. You can cite general theorems from the book if needed.

(1) [3 Pts] Suppose that P is a linear operator on an inner product space V satisfying $PP = P$ and

$$\langle Pv, w\rangle = \langle v, Pw\rangle \quad \text{for all } v, w \in V.$$

Prove that P is an orthogonal projection (HINT: need to show that the range of P is orthogonal to the kernel of P).

(2) [3 Pts] Suppose that (f_k) is a sequence of continuous functions defined on $\mathbb{R}^n$ and converging uniformly to a function f. Suppose that each f_k is bounded by a constant C_k. Prove that f is bounded.

(3)[3 Pts] Suppose that (f_k) is a sequence of continuous functions on $[0,1]$ and let $s_n(x) = \sum_{k=1}^{n} f_k(x)$. Show that if (s_n) converges uniformly on $[0,1]$ then the sequence (f_k) converges uniformly to 0.

(4)[3 Pts] For each of the following statements, either <u>prove</u> it (you can use the theorems discussed in class) or give a <u>counterexample</u> (in this case, you need to show how your counterexample disproves the statement).

(a) Suppose that (f_n) is a sequence of continuous functions on $[0,1]$ with $\lim_{n\to\infty} f_n = f$. Then $\lim \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$.

(b) Let f and g be uniformly continuous functions. Then the product $f\,g$ is a uniformly continuous function.

(c) A closed and bounded subset of a normed space is compact.

TEST #2 SOLUTION

① Since $PP = P$, then for $v, w \in V$

$$\langle Pv, (I-P)w\rangle = \langle Pv, w\rangle - \langle Pv, Pw\rangle = \cancel{\langle Pv, w\rangle}$$

(using hypothesis) $= \langle v, Pw\rangle - \langle v, PPw\rangle = \langle v, Pw\rangle - \langle v, Pw\rangle = 0$

This shows that $Pv \perp (I-P)w$ for any $v, w \in V$ proving that $\mathrm{ran}(P) \perp \ker(P)$.

RK $Px = 0$ iff $(I-P)x = x$ and $x = (I-P)y$ implies $Px = (P-P^2)y = 0$

This shows ~~that~~ $\ker(P) = \mathrm{ran}(I-P)$

② Given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $\|f_k - f\|_\infty < \varepsilon$ if $k \geq N$.

Since $\|f_k\|_\infty < C_k$, then $\|f\|_\infty \leq \|f - f_N\|_\infty + \|f_N\|_\infty < \varepsilon + C_N < \infty$,

~~where $C = \max\{C_k : 1 \leq k \leq N\}$~~

This shows that f is bounded.

③ Given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $\|S_n - S_m\|_\infty = \|\sum_{k=n+1}^{m} f_k\|_\infty < \varepsilon$ if $n, m \geq N$. We can choose $n = N$, $m = N+1$

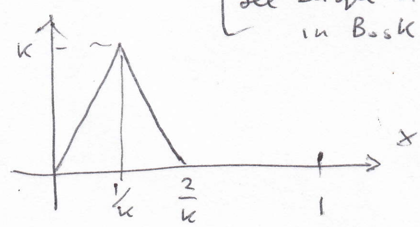
This IMPLIES that $\|f_k\|_\infty < \varepsilon$ if $k \geq N$

This shows that f_k converges UNIFORMLY TO 0

[See Example 8.1.2 in Book]

④ (a) FALSE

$$f_k(x) = \begin{cases} k^2 x & \text{if } 0 \leq x \leq 1/k \\ k^2\left(\frac{2}{k} - x\right) & \text{if } 1/k \leq x \leq \frac{2}{k} \\ 0 & \text{if } 2/k \leq x \leq 1 \end{cases}$$



$\lim_{k\to\infty} f_k(x) = 0$

However $\int_0^1 f_k(x)\,dx = 1 \quad \forall k$. Hence $\lim_{k\to\infty} \int_0^1 f_k(x)\,dx \neq \int_0^1 \lim_k f_k(x)\,dx$

(b) FALSE Take $f(x) = x$, $g(x) = x$. $f(x)g(x) = x^2$ is NOT UNIF. continuous on $\mathbb{R}$

(c) FALSE Take $F = \{f_k(x) = x^k : k \geq 1\} \subset C[0,1]$

F is a closed and bounded set. However we have shown that the sequence has no convergent subsequence in $C[0,1]$, hence the set is NOT COMPACT. [Cf. EXAMPLE 8.61 in Book]

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① IF $y \in \mathrm{Ran}\,P$, then $Py = y$. IF $x \in \ker(P)$, $Px = 0$

$$\langle x, y\rangle = \langle x, Py\rangle = \langle Px, y\rangle = 0$$

Hence $\mathrm{Ran}(P) \perp \ker(P)$