Math 4332/6313 – Spring 2017 Name:

Test #3

This is an open book test. Please, write clearly and justify all your steps, to get proper credit for your work. You can cite general theorems from the book if needed.

(1) [3 Pts] Suppose that f satisfies the hypotheses of Taylor's theorem at $x = a$.

(a) Show that $\lim_{x\to a} \frac{f(x)-P_n(x)}{(x-a)^n} = 0$, where P_n is the Taylor polynomial of order n of f at a.

(b) If Q is a polynomial of degree n and $\lim_{x\to a} \frac{f(x)-Q(x)}{(x-a)^n} = 0$, prove that $Q = P_n$.

(2) [3 Pts] The modulus of continuity of $f \in C[a,b]$ is defined for $\delta > 0$ as

$$\omega(f;\delta) = \sup\{|f(x_1) - f(x_2)| : |x_1 - x_2| < \delta,\ x_1, x_2 \in [a,b]\}.$$

(a) Show that $\omega(f;\delta_1) \leq \omega(f;\delta_2)$ if $\delta_1 \leq \delta_2$.

(b) If $f \in C^1[a,b]$, show that $\omega(f;\delta) \leq \|f'\|_\infty \, \delta$.

(3)[3 Pts] Construct an example of a differentiable map T from $\mathbb{R}$ to $\mathbb{R}$ whose fixed points are exactly the set of integers. Determine which fixed points are attracting or repelling.

(4)[3 Pts] Find the periodic points of the tripling map on the circle $T : \mathbb{T} \mapsto \mathbb{T}$ given by $T\theta = 3\theta$. Discuss which periodic points are attracting or repelling.

TEST #3 SOLUTION

(1) (a) $\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n}$ is an undeterminate form $\frac{0}{0}$.

Using Hopital rule, we get $\lim_{x \to a} \frac{f'(x) - P_n'(x)}{n(x-a)^{n-1}}$ which is also a form $\frac{0}{0}$

Repeating $n-1$ more times, we obtain $\lim_{x \to a} \frac{f^n(x) - P_n^n(x)}{n!} = 0$.

(b) $f(x) - Q(x) = f(x) - P_n(x) + P_n(x) - Q(x)$

By part (a), if $\lim_{x \to a} \frac{f(x) - Q(x)}{(x-a)^n} = 0$, then it must be

$$\lim_{x \to a} \frac{P_n(x) - Q(x)}{(x-a)^n} = \lim_{x \to a} \frac{f(x) - Q(x)}{(x-a)^n} - \lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$$

This implies that $\lim_{x \to a} P_n(x) - Q(x) = 0$, that is $\frac{P_n(x) - Q(x)}{(x-e)^n}$ is a form $\frac{0}{0}$ as $x \to a$

It must be $P_n(a) = Q(a)$, Applying Hopital rule repeatedly, we observe that it must be $P_n'(a) = Q'(a)$ and so forth $P_n^k(a) = Q^k(a)$ $\forall\, 0 \le k \le n$

This implies that $P_n(x) = Q(x)$ because they are both polynomials of degree n.

(2) (a) ~~the~~ If $\delta_1 < \delta_2$ then $\{|f(x_1) - f(x_2)| : |x_1 - x_2| < \delta_1 < \delta_2\} \subseteq \{|f(x_1) - f(x_2)| : |x_1 - x_2| < \delta_2\}$

Therefore ~~sup~~ $\omega(f; \delta_1) \le \omega(f; \delta_2)$

(b) By MVT, given any $x_1, x_2 \in [a, b]$, $\exists\, z \in (a, b)$ s.t. $f(x_1) - f(x_2) = f'(z)(x_1 - x_2)$

Therefore $|f(x_1) - f(x_2)| \le \|f'\|_\infty |x_1 - x_2|$

It follows that $\omega(f, \delta) = \sup\{|f(x_1) - f(x_2)| : |x_1 - x_2| < \delta\} \le \|f'\| \delta$

(3) Set $Tx = x + \sin \pi x$. T is differentiable.

Fixed points $Tx = x + \sin \pi x = x \iff \sin \pi x = 0 \iff x \in \mathbb{Z}$

Fixed points is exactly the set of INTEGERS

$T'(x) = 1 + \pi \cos \pi x$. For $x \in \mathbb{Z}$ $T'(n) = 1 + \pi \cos \pi n = 1 + \pi(-1)^n$

For any $n \in \mathbb{Z}$, $|T'(n)| > 1$, hence any ~~periodic~~ Fixed point is REPELLENT

(4) $T\theta = 3\theta$, $\theta \in \mathbb{T}$

A point θ is periodic of period n if $\theta = T^n \theta = 3^n \theta$ (mod 2π)

This is equivalent to $(3^n - 1)\theta = 2\pi k$ for $k \in \mathbb{Z}$.

Thus any point $\frac{2\pi}{3^n - 1}$ is periodic of period n

Any point $\frac{2\pi s}{3^n - 1}$ for $n \ge 1$ and $1 \le s \le 3^n - 1$ is periodic

Since $T'\theta = 3$, every periodic point is REPELLENT