HW1 SOLUTION

TOTAL POINTS =

(1) Let $\langle v, u \rangle = \bar{u} M v^T$ where $M = \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix}$

Notice that $M = \bar{M}^T$

TO SHOW THAT THIS IS AN INNER PRODUCT WE SHOW

- POSITIVITY (1.5 Pts)

$$\langle v, v \rangle = \bar{v} M v^T = (\bar{v}_1 \ \bar{v}_2) \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= 2|v_1|^2 - i \bar{v}_1 v_2 + i \bar{v}_2 v_1 + 3|v_2|^2$$

$$= 2|v_1|^2 + 3|v_2|^2 + 2\,\mathrm{Im}(\bar{v}_1 v_2)$$

Since $2|\mathrm{Im}(\bar{v}_1 v_2)| \le 2|\bar{v}_1 v_2| \le |v_1|^2 + |v_2|^2$, then

$$2|v_1|^2 + 3|v_2|^2 - 2\,\mathrm{Im}(\bar{v}_2 v_1) \ge 0$$

It is zero ~~[illegible]~~ iff $v = 0$

- LINEARITY (1.5 Pts)

$$\langle v + w, u \rangle = \bar{u} M (v+w)^T = \bar{u} M (v^T + w^T)$$

$$= \bar{u} M v^T + \bar{u} M w^T = \langle v, u \rangle + \langle w, u \rangle$$

- HOMOGENEITY (1.5 Pts)

$$\langle c v, u \rangle = \bar{u} M (cv)^T = \bar{u} M (c v^T) = c \bar{u} M v^T = c \langle v, u \rangle$$

- CONJUGATE SYMMETRY (1.5 Pts)

$$\langle v, u \rangle = \bar{u} M v^T$$

By the properties of DOT PRODUCT or MATRIX MULTIPLICATION, for 2 vectors a, b, $a b^T = a^T b$

Hence $\bar{u} M v^T = (M v^T)^T \bar{u}^T = v M^T \bar{u}^T = v \bar{M} \bar{u}^T = \overline{(\bar{v} M u^T)}$ (since $M^T = \bar{M}$) $= \overline{\langle u, v \rangle}$

This can also be proved directly, by computing the expression for $\bar{u} M v^T$ and $v \bar{M} \bar{u}^T$.

However the proof presented here is more general and quicker

HW1

2

- LINEARITY let $f_1, f_2, g \in L^2([a,b])$

1.5 Pt

$$\langle f_1 + f_2, g \rangle = \int_a^b (f_1(t) + f_2(t))\,\overline{g(t)}\,dt =$$

$$= \int_a^b f_1(t)\,\overline{g(t)}\,dt + \int_a^b f_2(t)\,\overline{g(t)}\,dt = \langle f_1, g \rangle + \langle f_2, g$$

- HOMOGENEITY let $c \in \mathbb{C}$.

1 Pt.

$$\langle c f, g \rangle = \int_a^b c\, f(t)\,\overline{g(t)}\,dt = c \int_a^b f(t)\,\overline{g(t)}\,dt = c \langle f, g \rangle$$

- CONJUGATE SYMMETRY

1.5 Pt

$$\langle f, g \rangle = \int_a^b f(t)\,\overline{g(t)}\,dt = \overline{\int_a^b \overline{f(t)}\,g(t)\,dt} =$$

$$= \overline{\langle g, f \rangle}$$