HW #1 SOLUTION

(3) $\langle V, W \rangle = (\overline{w_1}, \overline{w_2}) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let $v_1 = -2v_2$. Then $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2v_2 \\ v_2 \end{pmatrix} = -2v_2 + 2v_2 - 4v_2 + 4v_2 = 0$

This shows that $\langle V, V \rangle = 0$ for all $V = (v_1, v_2)$ when $v_1 = -2v_2$.
Positivity property is not satisfied. Hence $\langle V, W \rangle$ is NOT an INNER PRODUCT.

(4)
- $\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}\,dt = \overline{\int_a^b \overline{f(t)} g(t)\,dt} = \overline{\langle g, f \rangle}$ (conjugate symmetry)
- $\langle cf, g \rangle = \int_a^b c f(t)\overline{g(t)}\,dt = c \int_a^b f(t)\overline{g(t)}\,dt = c\langle f, g \rangle$ (homogeneity)
- $\langle f + h, g \rangle = \int_a^b (f(t) + h(t))\overline{g(t)}\,dt = \int_a^b f(t)\overline{g(t)}\,dt + \int_a^b h(t)\overline{g(t)}\,dt = \langle f, g \rangle + \langle h, g \rangle$ (linearity)
- (POSITIVITY) Suppose f is continuous on $[a,b]$.

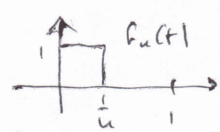
Arguing by contradiction, suppose $\exists\, t_0 \in [a,b]$ s.t. $|f(t_0)| > 0$ but $\int |f|^2 = 0$.
Since f is continuous, given $\varepsilon = \frac{|f(t_0)|}{2}$, there exists a $\delta > 0$ s.t.
$|f(t) - f(t_0)| < \frac{|f(t_0)|}{2}$, ~~this implies~~ for $t \in [t_0 - \delta, t_0 + \delta]$
This implies that $|f(t)| > \frac{|f(t_0)|}{2}$ for $t \in [t_0 - \delta, t_0 + \delta]$.
It follows that $\int_a^b |f|^2\,dt \geq \int_{t_0-\delta}^{t_0+\delta} \frac{|f(t_0)|^2}{4}\,dt = 2\delta \frac{|f(t_0)|^2}{4} = \frac{\delta}{2}|f(t_0)|^2 > 0$

This is a CONTRADICTION. It must be $f(t) = 0$. $\forall t \in [a,b]$.

(5) For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in $\mathbb{C}^n$, let $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$

- POSITIVITY. $\langle x, x \rangle = \sum_{i=1}^n |x_i|^2 = 0 \iff x_i = 0\ \forall i \iff x = 0$
- CONJUGATE SYMMETRY $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i} = \overline{\sum_{i=1}^n \overline{x_i} y_i} = \overline{\langle y, x \rangle}$
- homogeneity $\langle cx, y \rangle = \sum_{i=1}^n c x_i \overline{y_i} = c \sum_{i=1}^n x_i \overline{y_i} = c\langle x, y \rangle$
- linearity $\langle x + z, y \rangle = \sum_{i=1}^n (x_i + z_i)\overline{y_i} = \sum_{i=1}^n x_i \overline{y_i} + \sum_{i=1}^n z_i \overline{y_i} = \langle x, y \rangle + \langle z, y \rangle$

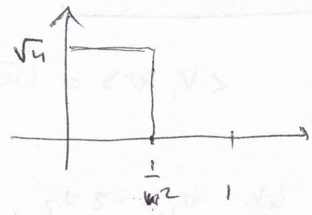
(6) $\| f_n - 0 \|_{L^2}^2 = \int_0^1 |f_n(t)|^2\,dt = \int_0^{1/n} dt = \frac{1}{n}$



Since $\lim_{n\to\infty} \| f_n - 0 \|_{L^2} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, then $f_n \to 0$ in $L^2[0,1]$

Note $f_n(0) = 1\ \forall n$.
Given ~~any~~ $\varepsilon = \frac{1}{2}$, there is no N s.t. $|f_n(0)| < \frac{1}{2}$.
Hence $f_n(t)$ does NOT converge to $f(t) = 0$ for all $t \in [0,1]$. Convergence is NOT UNIFORM.

(7) $\|f_n - 0\|_{L^2}^2 = \int_0^{1/n^2} |f_n(t)|^2\,dt = \int_0^{1/n^2} n\,dt = \frac{1}{n}$



$\lim_{n\to\infty} \|f_n\|_{L^2} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

This shows that ~~f~~ $f_n \to 0$ in $L^2[0,1]$

$f_n(0) = \sqrt{n}$ $\qquad \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \sqrt{n} = \infty$

This shows that $f_n(t)$ does NOT CONVERGE POINTWISE to 0.