HW3 SOLUTION

(1) (i) As done in class, the functions $\{1, \cos x, \sin x\}$ have to be normalized. They are orthogonal

Since $\int_{-\pi}^{\pi} \sin^2 x \, dx = \int_{-\pi}^{\pi} \cos^2 x \, dx = \pi$, then

[1Pt] $\underline{e_1(x) = \frac{1}{\sqrt{2\pi}}, \quad e_2(x) = \frac{1}{\sqrt{\pi}} \cos x, \quad e_3(x) = \frac{1}{\sqrt{\pi}} \sin x \quad \text{is an ONB of } V_0}$

(ii) $\int_{-\pi}^{\pi} \cos 2x \cos x \, dx = \int_{-\pi}^{\pi} \cos 2x \sin x \, dx = \int_{-\pi}^{\pi} \sin 2x \cos x = \int_{-\pi}^{\pi} \sin 2x \sin x \, dx = 0$

[1Pt] $\int_{-\pi}^{\pi} \cos 2x \, dx = \int_{-\pi}^{\pi} \sin 2x \, dx = 0$ As stated in class

Hence $V_0 \perp V_1$

(iii) $\cos 3x$ is orthogonal to V_0 by the prop. of trig functions.

Hence the ORTHOGONAL PROJECTION of $f(x) = \cos 3x$

[1Pt] onto V_0 is 0

(iv) ORTH PROJ onto V_0

$$f_{00}(x) = \cancel{\langle f, 1\rangle} + \cancel{\frac{1}{\pi}\langle f, \cos(\cdot)\rangle \cos x} + \frac{1}{\pi}\langle f, \sin(\cdot)\rangle \sin x$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} x \sin x \, dx\right) \sin x$$

(1.5Pt) $$\frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx \, dx = \frac{2}{\pi}\int_{0}^{\pi} x \sin kx \, dx = \frac{2}{\pi}\frac{\pi}{k}(-1)^{k+1} = \frac{2}{k}(-1)^{k+1}$$

(DONE IN CLASS

Hence $f_{00}(x) = 2 \sin x$

ORTH PROJ onto V_1

(1.5Pt) $$f_{01}(x) = \cancel{\langle f, \cos(2\cdot)\rangle \cos 2x} + \langle f, \sin(2\cdot)\rangle \sin 2x$$

$$= -\sin 2x$$

(2) As shown in class, φ, ψ are ORTHONORMAL

The ORTH PROJECTION of $f(x) = e^{-x}$ onto V_0 is

$$f_0(x) = \langle f, \varphi \rangle \varphi(x) + \langle f, \psi \rangle \psi(x)$$

[1 Pt] $$\langle f, \varphi \rangle = \int_0^1 e^{-x}\, dx = -e^{-x}\Big|_0^1 = 1 - e^{-1}$$

(1 Pt) $$\langle f, \psi \rangle = \int_0^{1/2} e^{-x}\, dx - \int_{\frac{1}{2}}^1 e^{-x}\, dx = -e^{-x}\Big|_0^{1/2} + e^{-x}\Big|_{\frac{1}{2}}^1$$

$$= \left(1 - e^{-\frac{1}{2}}\right) + e^{-1} - e^{-\frac{1}{2}}$$

$$= 1 - 2e^{-\frac{1}{2}} + e^{-1}$$

[1 Pt] $$f_0(x) = (1 - e^{-1})\varphi(x) + \left(1 + e^{-1} - 2e^{-\frac{1}{2}}\right)\psi(x)$$