HW #7 SOLUTION TOT Points = 10

(2)

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin 3t \, e^{-i\omega t} \, dt$$

Since $\sin 3t$ is an odd function, then only the sin-part of the transform contributes to the integral.

$$= \frac{-i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin 3t \sin \omega t \, dt$$

$$= -\frac{2i}{\sqrt{2\pi}} \int_{0}^{\pi} \sin 3t \sin \omega t \, dt$$

Recall $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

[3Pts]

$$= -\frac{\cancel{2}i}{\sqrt{2\pi}} \int_{0}^{\pi} \left[\cos(3t-\omega t) - \cos(3t+\omega t)\right] dt$$

$$= -\frac{i}{\sqrt{2\pi}} \left. \frac{\sin(3-\omega)t}{3-\omega} \right|_0^{\pi} - \left. \frac{\sin(3+\omega)t}{3+\omega} \right|_0^{\pi} = -\frac{i}{\sqrt{2\pi}} \frac{(3+\omega)\sin(3-\omega)\pi - (3-\omega)\sin(3+\omega)\pi}{9-\omega^2}$$

$\sin(3-\omega)\pi = -\sin(\omega\pi - 3\pi) = \sin \omega\pi$

$\sin(3+\omega\pi) = -\sin \omega\pi$

$$= -\frac{i}{\sqrt{2\pi}} \frac{2 \cdot 3 \sin \omega\pi}{9-\omega^2} = -\frac{3\sqrt{2}i}{\sqrt{\pi}} \frac{\sin \omega\pi}{9-\omega^2}$$

(4)

Suppose $f(t) = f(-t)$ even

$f(t) = \overline{f(t)}$ real

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

$$\overline{\hat{f}(\omega)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(t)} e^{i\omega t} \, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt$$

[2Pts)

~~Test~~

(replace $t \to -\tau$)

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(-\tau) e^{-i\omega\tau} (-d\tau)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-i\omega t} \, d\tau$$

$$= \hat{f}(\omega)$$

Suppose $f(t) = \overline{f(t)}$

$f(t) = -f(-t)$

Then
$$\overline{\hat{f}(\omega)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(t)}\, e^{+i\omega t}\, dt = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-t)\, e^{i\omega t}\, dt$$

[2Pts]

let $\tau = -t$
$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(\tau)\, e^{i\omega t}\, (-d\tau)$$
$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega t}\, d\tau$$
$$= -\hat{f}(\omega)$$

Hence $\hat{f}(\omega)$ is PURELY IMAGINARY

(6) $f_s(x) = \sqrt{s}\, e^{-sx^2}$

$$\hat{f}_s(\omega) = \sqrt{s}\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-sx^2} e^{-i\omega x}\, dx = \frac{\sqrt{s}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-s\left(x^2 + \frac{2i\omega x}{2s}\right)}\, dx$$
$$= \frac{\sqrt{s}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-s\left(x + \frac{i\omega}{2s}\right)^2} e^{-\frac{\omega^2}{4s}}\, dx$$

[3Pts]

$$= \frac{\sqrt{s}}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4s}} \int_{\mathbb{R}} e^{-\left(\sqrt{s}x + \frac{i\omega}{2\sqrt{s}}\right)^2}\, dx$$

let $y = \sqrt{s}x + \frac{i\omega}{2\sqrt{s}}$
$$= \frac{e^{-\frac{\omega^2}{4s}}}{\sqrt{2\pi}} \underbrace{\int_{\mathbb{R}} e^{-y^2}\, dy}_{\sqrt{\pi}} = \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4s}}$$

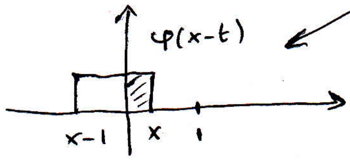
$dy = \sqrt{s}\, dx$

HW 8 SOLUTION

(5)

$$\varphi(t) = \chi_{[0,1]}(t)$$

$$\varphi * \varphi(x) = \int_{\mathbb{R}} \varphi(t)\,\varphi(x-t)\,dt = \int_0^1 \varphi(x-t)\,dt = \begin{cases} 0 & \text{if } x \geq 2 \\ 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 2-x & \text{if } 1 \leq x < 2 \end{cases}$$



$$= \boxed{\begin{cases} 1-|x-1| & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}}$$

(10)

$$h(t) = \begin{cases} A e^{-\alpha t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{h}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(t)\, e^{-i\omega t}\, dt = \frac{A}{\sqrt{2\pi}} \int_0^\infty e^{-(\alpha+i\omega)t}\, dt = \frac{A}{\sqrt{2\pi}} \left. \frac{-e^{-(\alpha+i\omega)t}}{\alpha+i\omega} \right|_0^\infty$$

$$= \boxed{\frac{A}{\sqrt{2\pi}\,(\alpha+i\omega)}}$$

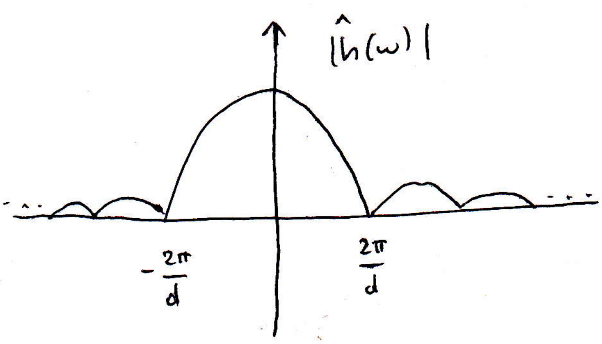
(12)

$$h(t) = \frac{1}{d}\chi_{[0,d]}(t) \qquad \hat{h}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1}{d}\int_0^d e^{-i\omega t}\, dt = \frac{i}{\sqrt{2\pi}\, d\omega}\left(e^{-i\omega d} - 1\right)$$

$$= \frac{1}{\sqrt{2\pi}\, d\omega}\left(\sin(\omega d) + i(\cos \omega d - 1)\right)$$

$$|\hat{h}(\omega)| = \frac{1}{\sqrt{2\pi}\, d|\omega|}\sqrt{\sin^2(\omega d) + (\cos(\omega d)-1)^2} = \frac{1}{\sqrt{\pi}}\frac{\sqrt{1-\cos(\omega d)}}{|\omega| d}$$

$$= \frac{\sqrt{2}\,|\sin(\omega \frac{d}{2})|}{\sqrt{\pi}\,|\omega| d} = \frac{1}{\sqrt{2\pi}}\left|\operatorname{sinc}\left(\frac{\omega d}{2}\right)\right|$$



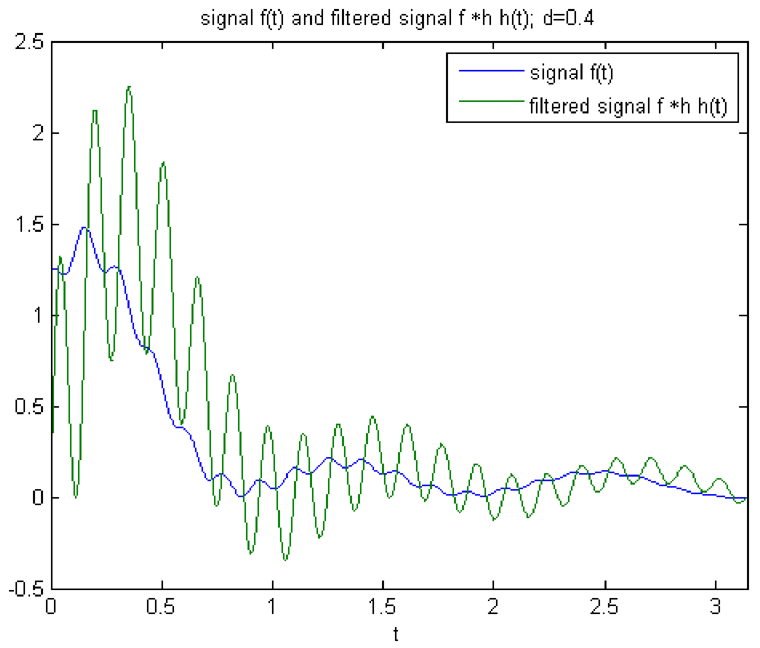
First zero at $\frac{\omega d}{2} = \pi \Rightarrow \omega = \frac{2\pi}{d}$

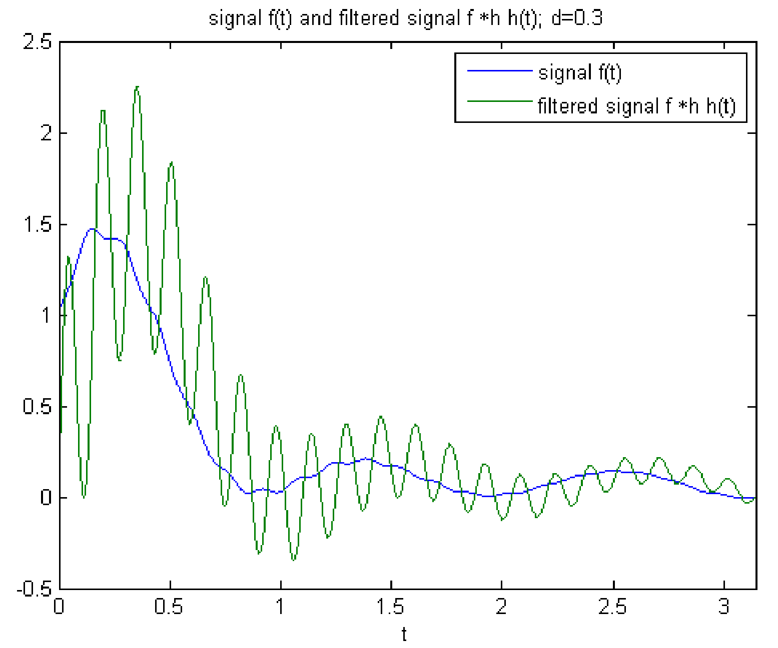
To retain frequencies ≤ 5, set

$$5 \leq \frac{\pi}{d} \Rightarrow \boxed{d \leq \frac{\pi}{5} \simeq 0.6}$$

To remove frequencies ≥ 30 set

$$30 \geq \frac{2\pi}{d} \Rightarrow \boxed{d \geq \frac{2\pi}{30} \simeq 0.2}$$






```
MATLAB CODE
% Define time range
t=0:0.0001:pi;
% Define d
d=0.4;
% Define f
f = exp(-t).*(sin(t)+sin(3*t)+sin(5*t)+sin(40*t));
% Define h. Note that its area should be one.
% To achieve this, I divide by its length.
norm=sum(t>=0 & t<=d);
hd = (t>=0 & t<=d)./norm+(t>d).*0;
ffiltered = conv(f,hd);
% The convolved signal has support longer than the support of f.
% I choose a window corresponding to the locations where the filter
% support is contained inside the support of f
len=length(t);
plot(t,ffiltered(norm:len+norm-1),t,f)
title('signal f(t) and filtered signal f \asth h(t); d=0.4');
axis([0 pi -0.5 2.5]);xlabel('t');
legend('signal f(t)','filtered signal f\asth h(t)')
```