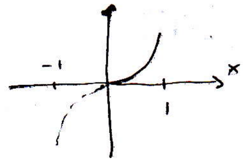
HW #5 SOLUTION

(4) Expand $f(x) = x^2$, $0 \le x \le 1$, into a <u>SINE SERIES</u>

[3Pts]



$$F(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

$$b_k = 2\int_0^1 x^2 \sin(k\pi x)\,dx = 2\left[-\frac{x^2}{k\pi}\cos(k\pi x)\Big|_0^1 + \int_0^1 \frac{2x}{k\pi}\cos(k\pi x)\,dx\right]$$

$$= 2\left[-\frac{1}{k\pi}\cos(k\pi) + \cancel{\frac{2x}{k^2\pi^2}\sin(k\pi x)\Big|_0^1} - \frac{2}{k^2\pi^2}\int_0^1 \sin(k\pi x)\,dx\right]$$

$$= 2\left[-\frac{1}{k\pi}\cos(k\pi) + \frac{2}{k^3\pi^3}\cos(k\pi x)\Big|_0^1\right]$$

$$= -\frac{2}{k\pi}\cos(k\pi) + \frac{4}{k^3\pi^3}\cos k\pi - \frac{4}{k^3\pi^3} = \boxed{-\frac{2(-1)^k}{k\pi} + \frac{4}{k^3\pi^3}\left((-1)^k - 1\right)}$$

(18) $f(x) = \sum_n \alpha_n e^{inx}$, $g(x) = \sum_m \beta_m e^{imx}$

[2Pts]

$$f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,g(x-t)\,dt =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_n \alpha_n e^{int} \sum_m \beta_m e^{im(x-t)}\,dt =$$

$$= \frac{1}{2\pi}\sum_n \alpha_n \sum_m \beta_m e^{imx} \underbrace{\int_{-\pi}^{\pi} e^{i(n-m)t}\,dt}_{=2\pi \text{ if } m=n,\ =0 \text{ otherwise}}$$

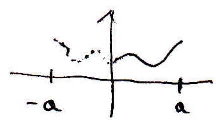
$$= \boxed{\sum_n \alpha_n \beta_n e^{inx}}$$

(20) Since the function is $2a$-periodic, to check continuity it is sufficient to examine the function over $[-a, a]$, including the end point a.

[4Pts]

<u>CASE 1</u> EVEN EXTENSION

$$f_e(x) = \begin{cases} f(x) & 0 \le x \le a \\ f(-x) & -a \le x < 0 \end{cases}$$



f is continuous on $(-a, 0) \cup (0, a)$

$\lim_{x \to 0^+} f_e(x) = f(0)$ $\qquad \lim_{x \to 0^-} f_e(x) = f(-0) = f(0)$. $\qquad$ Thus f_e is continuous at $x=0$

$\lim_{x \to a^-} f_e(x) = f(a)$ $\qquad \lim_{x \to a^+} f_e(x) = f(-(-a)) = f(a)$ $\qquad$ Thus f_e is continuous at $x = a$

<u>CASE 2</u> ODD EXTENSION

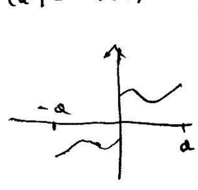
$$f_o(x) = \begin{cases} f(x) & 0 \le x \le a \\ -f(-x) & -a \le x < 0 \end{cases}$$

As above, we only need to check $x=0$ and $x=a$

$\lim_{x \to 0^+} f_o(x) = f(0)$. $\qquad \lim_{x \to 0^-} f_o(x) = -f(-0) = -f(0) \Rightarrow$ <u>NEED</u> $f(0) = -f(0) \Rightarrow f(0) = 0$

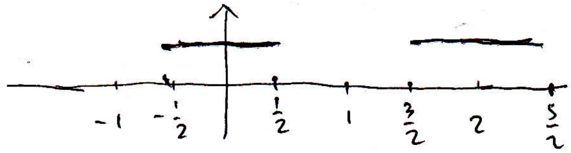
$\lim_{x \to a^-} f_o(x) = f(a)$ $\qquad \lim_{x \to a^+} f_o(x) = -f(-(-a)) = -f(a) \Rightarrow$ <u>NEED</u> $f(a) = -f(a) \Rightarrow f(a) = 0$

Continuity of ODD EXTENSION REQUIRES $f(0) = 0$, $f(a) = 0$



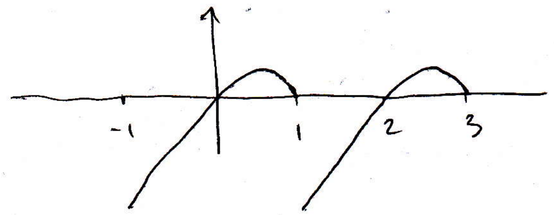
(23) [3 Pts]

(b) $f(x) = \begin{cases} 1 & x \in [-\frac{1}{2}, \frac{1}{2}] \\ 0 & x \in (-1,1) \setminus [-\frac{1}{2}, \frac{1}{2}] \end{cases}$



The periodization of f is discontinuous.
Thus the Fourier SERIES of f DOES NOT CONVERGE POINTWISE

(c) $f(x) = x - x^2, \quad x \in (-1,1)$



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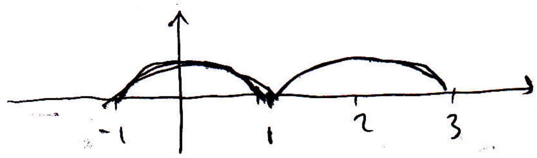
(d) $f(x) = 1 - x^2, \quad x \in (-1,1)$



The periodization of f is continuous, and the derivative is defined at all x.
This the Fourier SERIES of f CONVERGES UNIFORMLY

(26) [4 Pts]

- Suppose f real valued and even on $[-\pi,\pi]$ $\rightarrow$ $f(x) = f(-x)$, $f(x) = \overline{f(x)}$

F-coefficients: $\alpha_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx$ ↓ f real

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\, e^{-ikx}\,dx = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{ikx}\,dx} \qquad \text{let } x' = -x$$

$$= \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(-x') e^{-ikx'}\,dx'} = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x') e^{-ikx'}\,dx'} = \overline{\alpha_k}$$

f even

This shows α_k is REAL

- Suppose f is real valued and odd on $[-\pi,\pi]$. $\rightarrow$ $f(x) = -f(-x)$, $f(x) = \overline{f(x)}$

F. coefficients $\alpha_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx$

↓ f real

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\, e^{-ikx}\,dx = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{ikx}\,dx} \qquad \text{let } x' = -x$$

$$= \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(-x') e^{-ikx'}\,dx'} = -\overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x') e^{-ikx'}\,dx'} =$$

f odd

$$= -\overline{\alpha_k}$$

This shows that α_k is IMAGINARY