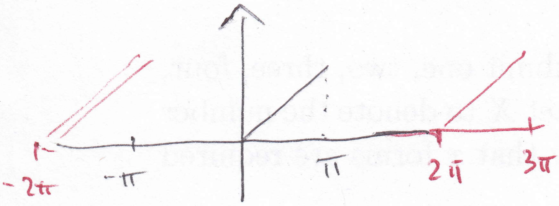
HW #6

(1) $f(x) = x, \quad x \in [0, \pi]$

(a) Fourier series of f valid in $(-\pi, \pi]$



$$F(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{1}{2\pi}\left.\frac{x^2}{2}\right|_0^{\pi} = \frac{\pi}{4}$$

$$a_k = \frac{1}{\pi}\int_0^{\pi} x\cos kx\,dx = \frac{1}{\pi}\left[\cancel{\left.\frac{x}{k}\sin kx\right|_0^{\pi}} - \int_0^{\pi}\frac{\sin kx}{k}\,dx\right]$$

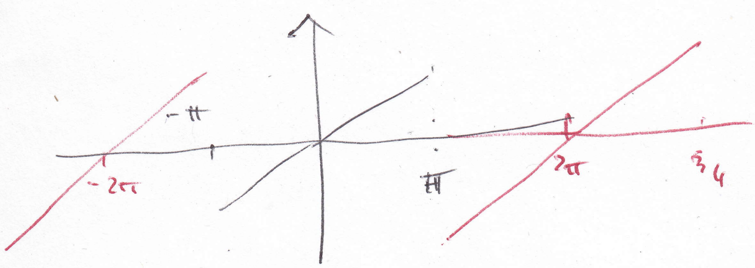
$$= -\frac{1}{k\pi}\left(-\left.\frac{\cos kx}{k}\right|_0^{\pi}\right) = \frac{1}{k^2\pi}\left(\cos k\pi - 1\right)$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} x\sin kx\,dx = \frac{1}{\pi}\left[-\left.\frac{x}{k}\cos kx\right|_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos kx\,dx\right]$$

$$= \frac{1}{k\pi}\left[-\pi\cos k\pi + \cancel{\left.\frac{1}{k^2}\sin kx\right|_0^{\pi}}\right] = \frac{-1}{k}(\cos k\pi)$$

$F(x)$ converges to f uniformly on $(-\pi, \pi)$, it does to converge to f at $\pm\pi$

(b) Sine series of f valid in $(-\pi, \pi]$



It is the same as the Fourier Series of x on $[-\pi, \pi]$

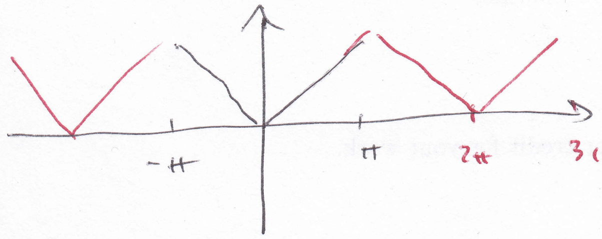
$$F(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

$$b_k = \frac{2}{\pi}\int_0^{\pi} x\sin kx\,dx = \frac{2}{\pi}\left(-\left.\frac{x}{k}\cos kx\right|_0^{\pi} + \cancel{\frac{1}{k}\int_0^{\pi}\cos kx\,dx}\right)$$

$$= -\frac{2}{k}\cos k\pi x = \frac{2}{k}(-1)^{k+1}$$

$F(x)$ converges unif to f on $(-\pi, \pi)$; it does NOT converge to f at $\pm\pi$

(c) Cosine series of f valid in $-\pi, \pi$



$$F(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx$$

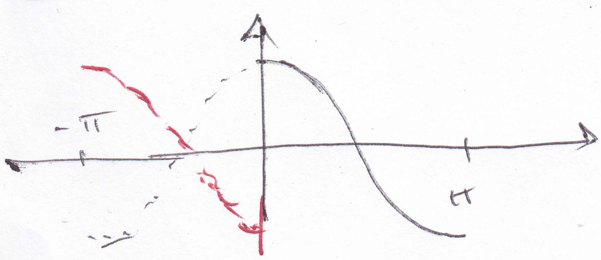
$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\, dx = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} x \cos kx\, dx = \frac{2}{\pi}\left[\cancel{\frac{x}{k}\sin kx \Big|_0^{\pi}} - \frac{1}{k}\int_0^{\pi} \sin kx\, dx \right]$$

$$= -\frac{2}{k\pi}\left(-\frac{1}{k}\cos kx \right)\Big|_0^{\pi} = \frac{2}{\pi k^2}\left(\cos k\pi - 1 \right)$$

$F(x)$ converges to f uniformly on $[-\pi, \pi]$.

(2) $f(x) = \cos x, \quad x \in [0, \pi]$



The cosine series of $f(x)$ valid in $(-\pi, \pi)$ is the Fourier series of $\cos x$

(a) $F(x) = \cos x$

(b) Sine series of $f(x)$:

$$F(x) = \sum_{k=1}^{\infty} b_k \sin kx.$$

$$b_k = \frac{2}{\pi}\int_0^{\pi} \cos x \sin kx\, dx$$

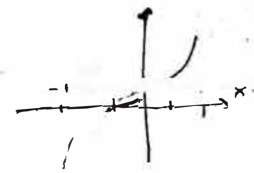
(3) $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

This expression is also the Fourier series.

It converges uniformly to $f(x)$ on $[-\pi, \pi]$

SOLUTION

(4) Expand $f(x) = x^2, \quad 0 \le x \le 1$, into a SINE SERIES



[3Pts]

$$F(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

$$b_k = 2\int_0^1 x^2 \sin(k\pi x)\,dx = 2\left[-\frac{x^2}{k\pi}\cos(k\pi x)\Big|_0^1 + \int_0^1 \frac{2x}{k\pi}\cos(k\pi x)\,dx\right]$$

$$= 2\left[-\frac{1}{k\pi}\cos(k\pi) + \cancel{\frac{2x}{k^2\pi^2}\sin(k\pi x)\Big|_0^1} - \frac{2}{k^2\pi^2}\int_0^1 \sin(k\pi x)\,dx\right]$$

$$= 2\left[-\frac{1}{k\pi}\cos(k\pi) + \frac{2}{k^3\pi^3}\cos(k\pi x)\Big|_0^1\right]$$

$$= -\frac{2}{k\pi}\cos(k\pi) + \frac{4}{k^3\pi^3}\cos k\pi - \frac{4}{k^3\pi^3} = \boxed{-\frac{2(-1)^k}{k\pi} + \frac{4}{k^3\pi^3}\left((-1)^k - 1\right)}$$

(18) [2Pts]

$$f(x) = \sum_n \alpha_n e^{inx}, \quad g(x) = \sum_m \beta_m e^{imx}$$

$$f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, g(x-t)\,dt =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_n \alpha_n e^{int} \sum_m \beta_m e^{im(x-t)}\,dt =$$

$$= \frac{1}{2\pi}\sum_n \alpha_n \sum_m \beta_m e^{imx} \underbrace{\int_{-\pi}^{\pi} e^{i(n-m)t}\,dt}_{=2\pi \text{ if } m=n, \quad =0 \text{ otherwise}}$$

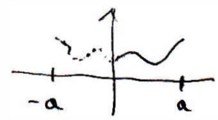
$$= \boxed{\sum_n \alpha_n \beta_n e^{inx}}$$

(20) [4Pts]

Since the function is $2a$-periodic, to check continuity it is sufficient to examine the function over $[-a, a]$, including the end point a.

CASE 1 EVEN EXTENSION

$$f_e(x) = \begin{cases} f(x) & 0 \le x \le a \\ f(-x) & -a \le x < 0 \end{cases}$$



f is continuous on $(-a, 0) \cup (0, a)$

$\lim_{x\to 0^+} f_e(x) = f(0)$ ~~$f(0)$~~ $\quad \lim_{x \to 0^-} f_e(x) = f(-0) = f(0)$. Thus f_e is continuous at $x=0$

$\lim_{x \to a^-} f_e(x) = f(a) \quad \lim_{x \to a^+} f_e(x) = f(-(-a)) = f(a)$ Thus f_e is continuous at $x = a$

CASE 2 ODD EXTENSION

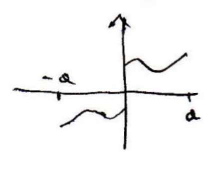
$$f_o(x) = \begin{cases} f(x) & 0 \le x \le a \\ -f(-x) & -a \le x < 0 \end{cases}$$

As above, we only need to check $x=0$ and $x=a$

$\lim_{x \to 0^+} f_o(x) = f(0). \quad \lim_{x \to 0^-} f_o(x) = -f(-0) = -f(0) \Rightarrow$ NEED $f(0) = -f(0) \Rightarrow f(0)$

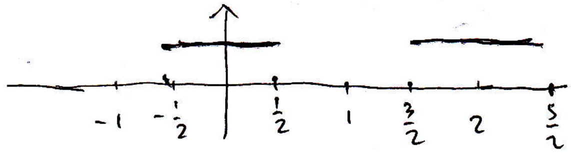
$\lim_{x \to a^-} f_o(x) = f(a) \quad \lim_{x \to a^+} f_o(x) = -f(-(-a)) = -f(a) \Rightarrow$ NEED $f(a) = -f(a) \Rightarrow f(a) =$

Continuity of ODD EXTENSION REQUIRES $f(0) = 0, \ f(a) = 0$



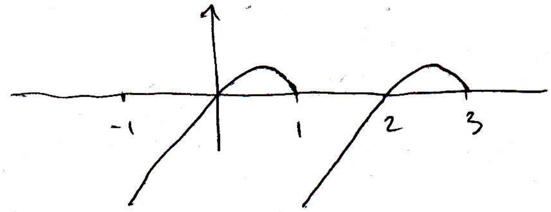
(23) [3 Pts]

(b) $f(x) = \begin{cases} 1 & x \in [-\frac{1}{2}, \frac{1}{2}] \\ 0 & x \in (-1,1) \setminus [-\frac{1}{2}, \frac{1}{2}] \end{cases}$



The periodization of f is discontinuous.
Thus the Fourier SERIES of f DOES NOT CONVERGE POINTWISE

(c) $f(x) = x - x^2, \quad x \in (-1,1)$



The periodization of f is DISCONTINUOUS.
Thus the Fourier series of f DOES NOT CONVERGE POINTWISE

(d) $f(x) = 1 - x^2, \quad x \in (-1,1)$

The periodization of f is continuous, and the derivative is defined at all x.
Thus the Fourier SERIES of f CONVERGES UNIFORMLY

(26) [4 Pts]

- Suppose f real valued and even on $[-\pi, \pi]$ $\to$ $f(x) = f(-x)$, $f(x) = \overline{f(x)}$

F-coefficients:

$$\alpha_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx \quad \downarrow f \text{ real}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} e^{-ikx} \, dx = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} \, dx} \qquad \text{let } x' = -x$$

$$= \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(-x') e^{-ikx'} \, dx'} = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') e^{-ikx'} \, dx'} = \overline{\alpha_k}$$

f even

This shows α_k is REAL

- Suppose f is real valued and odd on $[-\pi, \pi]$. $\to$ $f(x) = -f(-x)$, $f(x) = \overline{f(x)}$

F. coefficients

$$\alpha_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx$$

$\downarrow f$ real

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} e^{-ikx} \, dx = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} \, dx} \qquad \text{let } x' = -x$$

$$= \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(-x') e^{-ikx'} \, dx'} = -\overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') e^{-ikx'} \, dx'} =$$

f odd

$$= -\overline{\alpha_k}$$

This shows that α_k is IMAGINARY