HW #7 SOLUTION Tot Points = 10

(2)

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin 3t \, e^{-i\omega t} \, dt$$

Since $\sin 3t$ is an odd function, then only the sin-part of the transform contributes to the integral.

$$= \frac{-i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin 3t \sin \omega t \, dt$$

$$= -\frac{2i}{\sqrt{2\pi}} \int_{0}^{\pi} \sin 3t \sin \omega t \, dt$$

Recall $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

[3Pts]

$$= -\frac{\cancel{2}i}{\sqrt{2\pi}} \int_{0}^{\pi} \left[\cos(3t-\omega t) - \cos(3t+\omega t)\right] dt$$

$$= -\frac{i}{\sqrt{2\pi}} \left. \frac{\sin(3-\omega)t}{3-\omega} \right|_0^{\pi} - \left. \frac{\sin(3+\omega)t}{3+\omega} \right|_0^{\pi} = -\frac{i}{\sqrt{2\pi}} \frac{(3+\omega)\sin(3-\omega)\pi - (3-\omega)\sin(3+\omega)\pi}{9-\omega^2}$$

$\sin(3-\omega)\pi = -\sin(\omega\pi - 3\pi) = \sin \omega\pi$

$\sin(3+\omega\pi) = -\sin \omega\pi$

$$= -\frac{i}{\sqrt{2\pi}} \frac{2\cdot 3 \sin \omega\pi}{9-\omega^2} = -\frac{3\sqrt{2}\, i}{\sqrt{\pi}} \frac{\sin \omega\pi}{9-\omega^2}$$

(4)

Suppose $f(t) = f(-t)$ even

$f(t) = \overline{f(t)}$ real

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

$$\overline{\hat{f}(\omega)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(t)} e^{i\omega t} \, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt$$

[2Pts]

$$(\text{replace } t \to -\tau) \quad = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(-\tau) e^{-i\omega\tau} (-d\tau)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-i\omega t} \, d\tau$$

$$= \hat{f}(\omega)$$

Suppose $\overline{f(t)} = f(t)$

$f(t) = -f(-t)$

Then
$$\overline{\hat{f}(w)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(t)}\, e^{+iwt}\, dt = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-t)\, e^{iwt}\, dt$$

[2Pts]

let $\tau = -t$
$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(\tau)\, e^{iwt}\, (-d\tau)$$
$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\, e^{-iwt}\, d\tau$$
$$= -\hat{f}(w)$$

Hence $\hat{f}(w)$ is PURELY IMAGINARY

(6) $f_s(x) = \sqrt{s}\, e^{-sx^2}$

$$\hat{f}_s(w) = \sqrt{s}\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-sx^2} e^{-iwx}\, dx = \frac{\sqrt{s}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-s\left(x^2 + \frac{2iwx}{2s}\right)}\, dx$$
$$= \frac{\sqrt{s}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-s\left(x + \frac{iw}{2s}\right)^2} e^{-\frac{w^2}{4s}}\, dx$$

[3Pts]

$$= \frac{\sqrt{s}}{\sqrt{2\pi}} e^{-\frac{w^2}{4s}} \int_{\mathbb{R}} e^{-\left(\sqrt{s}x + \frac{iw}{2\sqrt{s}}\right)^2}\, dx$$

let $y = \sqrt{s}x + \frac{iw}{2\sqrt{s}}$
$$= \frac{e^{-\frac{w^2}{4s}}}{\sqrt{2\pi}} \underbrace{\int_{\mathbb{R}} e^{-y^2}\, dy}_{\sqrt{\pi}} = \frac{1}{\sqrt{2}} e^{-\frac{w^2}{4s}}$$

$dy = \sqrt{s}\, dx$