Math 4355 – Spring 2021 Name: SOLUTION

Quiz #4

Please, type or write legibly, scan, save file as LASTNAME_FIRSTNAME_Q4.pdf and email to `dlabate@math.uh.edu` or `dlabate@uh.edu`. You need to email to me no later than 11:30AM on Jan 28.

Consider the inner product space $V = L^2([0,1])$.

(1) [8 Pts] Compute the orthogonal projection of the function $f(x) = \sin(2\pi x)$, for $x \in [0,1]$, onto the subspace V_0 of V defined by $V_0 = \text{span}\{\phi, \psi\}$, where

$$\phi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution.

$\{\phi, \psi\}$ is an ONB of V_0.

$$\langle f, \phi \rangle = \int_0^1 \sin(2\pi x)\, dx = 0$$

$$\langle f, \psi \rangle = \int_0^{1/2} \sin(2\pi x)\, dx - \int_{1/2}^1 \sin(2\pi x)\, dx = \frac{1}{2\pi} \left(-\cos(2\pi x)|_0^{1/2} + \cos(2\pi x)|_{1/2}^1 \right) = \frac{2}{\pi}$$

Hence the orthogonal projection of f onto V_1 is

$$f_1 = \langle f, \phi \rangle \phi + \langle f, \psi \rangle \psi = \frac{2}{\pi}\, \psi$$

(2) [3 Pts] Let $V_1 = \operatorname{span}\{\psi_2\} \subset V$ where $\psi_2(x) = \psi(2x)$. Show that $V_1 \perp V_0$, that is, V_1 is orthogonal to V_0.

[Hint: plot ψ_2; this will help you to guide your calculation showing that ψ_2 is orthogonal to both ϕ and ψ.]

Solution.

$$\psi_2(x) = \psi(2x) = \begin{cases} 1 & 0 \leq x < \frac{1}{4} \\ -1 & \frac{1}{4} \leq x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Since both ϕ and ψ are constant and equal to 1 in $[0, \frac{1}{2}]$, then

$$\langle \psi_2, \phi \rangle = \int_0^{1/2} \psi(2x)\, dx = \int_0^{1/4} dx - \int_{1/4}^{1/2} dx = \frac{1}{4} - \frac{1}{4} = 0$$

$$\langle \psi_2, \psi \rangle = \int_0^{1/2} \psi(2x)\, dx = \int_0^{1/4} dx - \int_{1/4}^{1/2} dx = \frac{1}{4} - \frac{1}{4} = 0$$