Math 4355 – Spring 2021 Name:

FINAL EXAM version 1 (Total points = 31 + 2 extra credit)

Please, write legibly and justify all your steps to get credit for your work. To submit your exam, scan it, save it as a PDF file named LASTNAME_FIRSTNAME_exam.pdf (be sure your file size is below 5MB) and email to `dlabate@math.uh.edu` *or* `dlabate@uh.edu`. *NOTE: You need to send your email before 1:10 PM on May 11 to receive credit.*

- (1)[8 Pts] Let

$$f(x) = \begin{cases} 1 & 0 \le x \le 1/2 \\ -1 & 1/2 < x \le 1 \\ 0 & \text{otherwise .} \end{cases}$$

(a) Compute the Fourier sine series of f on the interval $[0, 1]$. Please, simplify your solution to obtain an explicit formula for the Fourier coefficients.

(b) Compute the Fourier cosine series of f on the interval $[0, 1]$. Please, simplify your solution to obtain an explicit formula for the Fourier coefficients.

(c) Sketch the plots of the Fourier sine series of f and Fourier cosine series of f valid over the interval $[-1, 1]$ and answer the following question: Do the Fourier sine and cosine series of f converge uniformly to f on the real line, $\infty < x < \infty$? Justify your answer.

- (2)[3 Pts] Let $f(x) = (\cos x + 1)\cos x$.

(a) Expand the function $f(x)$ in a Fourier series valid on the interval $[-\pi, \pi]$. (HINT: Note the special form of f. There is no need to solve any integration.).

(b) Does the Fourier series of f converge uniformly to f on the real line, $\infty < x < \infty$? Justify your answer.

- (3)[4 Pts] Let $f(t) = \begin{cases} |t| & -\frac{1}{2} \le t < \frac{1}{2} \\ 0 & \text{otherwise .} \end{cases}$

Using the definition, compute the Fourier transform of f.
[HINT: Write $e^{-i\omega t} = \cos \omega t - i \sin \omega t$, then take advantage of the symmetry of f.]

- (4)[6 Pts] Let

$$\phi(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{otherwise ,} \end{cases} \quad \text{and} \quad g(t) = \begin{cases} t & 0 \le t < 3 \\ 0 & \text{otherwise .} \end{cases}$$

(a) Compute $h(x) = (\phi * g)(x)$.
(b) Sketch the graphs of h over the interval $[-1, 6]$.

- (5) [6 Pts] Let ϕ and ψ be the Haar scaling and wavelet functions respectively. Let V_j and W_j be the spaces generated by $\phi(2^j x - k)$, $k \in \mathbb{Z}$, and $\psi(2^j x - k)$, $k \in \mathbb{Z}$, respectively. Consider the function defined on $0 \le x < 1$ given by

$$f(x) = \begin{cases} 2 & 0 \le x < 1/8 \\ 0 & 1/8 \le x < 2/8 \\ 2 & 2/8 \le x < 3/8 \\ 2 & 3/8 \le x < 4/8 \\ 3 & 4/8 \le x < 5/8 \\ 1 & 5/8 \le x < 6/8 \\ -1 & 6/8 \le x < 7/8 \\ -3 & 7/8 \le x < 1. \end{cases}$$

(a) Express f in terms of the basis for V_3.

(b) Find the discrete Haar wavelet decomposition of f. That is, decompose f into its component parts in V_0, W_0, W_1 and W_2.

(c) Express the Haar wavelet decomposition of f in terms of the basis functions of V_0, W_0, W_1 and W_2.

- (6) [4 Pts] Let $g \in L^2(\mathbb{R})$ be a function such that

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \chi_{[-\pi,\pi]}(\xi).$$

(a) Letting $g_{j,k}(x) = 2^{j/2} g(2^j x - k)$, use the properties of the Fourier transform to compute $\hat{g}_{j,k}(\xi)$, that is, the Fourier transform of

$$g_{j,k}(x) = 2^{j/2} g(2^j x - k)$$

(b) Verify that $\|g\|_2 = 1$ and shows that $\|g_{j,k}\|_2 = \|g\|_2$

- (7)[2 Pts - Extra Credit] Consider the filter

$$f \to f * h_d,$$

where

$$h_d(t) = \begin{cases} 1/d & 0 \le t < d \\ 0 & \text{otherwise .} \end{cases}$$

Let

$$f(t) = e^{-t} \left(\cos 3t - \sin 13t + \cos 50t + 3 \sin 130t \right), \quad t \in [0, 2\pi].$$

Which value(s) of the parameter d for the filter h_d will ensure that the components of the signal f with frequencies above 125 are removed and the frequencies in the range 0 to 50 are retained? Justify your answer.