HW 2 SOLUTION

Ex 9 The orthogonal complement of $v=(1,-2,1)$ in $\mathbb{R}^3$ is the plane of equation $x-2y+z=0$.

Note this plane is $V=\text{span}\{(2,1,0),(1,0,-1)\}$

Ex 10 Let $V=L^2[0,1]$ and $f=1$ on $[0,1]$.

Let $V_0=\text{span}\{f\}=\{f\in V:\ f \text{ is constant}\}$.

$V_0^\perp=\{g\in V:\ \langle f,g\rangle=\int_0^1 g(x)\,dx=0\}$. This is the set of functions with avg $=0$

Ex 11 $V=L^2[0,\pi]$

Suppose $\langle f,\cos t\rangle=0$

then $\langle f',\sin t\rangle=\int_0^\pi f'(t)\sin t\,dt=\cancel{f(t)\sin(t)\Big|_0^\pi}-\int_0^\pi f(t)\cos t\,dt=-\langle f,\cos t\rangle=0$

[Prob 4] $V=L^2[-\pi,\pi]$

$V_0=\text{span}\{1,\cos x,\sin x\}$ We found that $\{1,\cos x,\sin x\}$ is ON set

(i) $\left\{\frac{1}{\sqrt{2\pi}},\frac{1}{\sqrt{\pi}}\cos x,\frac{1}{\sqrt{\pi}}\sin x\right\}$ is an ON set. It has same span as $\{1,\cos x,\sin x\}$

Hence it is an ONB of V_0

(ii) $\langle 1,\cos 2x\rangle=\int_{-\pi}^{\pi}\cos 2x\,dx=0$, $\langle 1,\sin 2x\rangle=\int_{-\pi}^{\pi}\sin 2x\,dx=0$

$\langle\cos x,\cos 2x\rangle=\int_{-\pi}^{\pi}\cos x\cos 2x\,dx=\frac{1}{2}\int_{-\pi}^{\pi}(\cos 3x+\cos x)\,dx=\frac{1}{2}\left[\int_{-\pi}^{\pi}\cos 3x\,dx+\int_{-\pi}^{\pi}\cos x\,dx\right]=0$

$\langle\sin x,\cos 2x\rangle=\int_{-\pi}^{\pi}\cos 2x\sin x\,dx=\int_{-\pi}^{\pi}(2\cos^2x-1)\sin x\,dx\ \underset{u=\cos x}{=}\ \int_{-1}^{-1}(1-2u^2)\,du=0$

$\langle\cos x,\sin 2x\rangle=\int_{-\pi}^{\pi}\cos x\sin 2x\,dx=\int_{-\pi}^{\pi}2\cos^2x\sin x\,dx\ \underset{u=\cos x}{=}\ -\int_{-1}^{-1}2u^2\,du=0$

$\langle\sin x,\sin 2x\rangle=\int_{-\pi}^{\pi}\sin x\sin 2x\,dx=\int_{-\pi}^{\pi}2\sin^2x\cos x\,dx\ \underset{u=\sin x}{=}\ \int_0^0 2u^2\,du=0$

This shows that $V_1\perp V_0$

V_1 is NOT the orthogonal complement of V_0 since there are other functions in V that are orthogonal to $\{1,\cos x,\sin x\}$. For example $f(x)=\cos 3x$, see part (iii)

(iii) $\langle 1,\cos 3x\rangle=0$ $\langle\cos x,\cos 3x\rangle=\int_{-\pi}^{\pi}\cos x\cos 3x\,dx=\frac{1}{2}\int_{-\pi}^{\pi}(\cos 4x+\cos x)\,dx=0$

$\langle\sin x,\cos 3x\rangle=\frac{1}{2}\int_{-\pi}^{\pi}(\sin 4x-\sin 2x)\,dx=0$

This shows that $\cos 3x\perp\{1,\sin x,\cos x\}$

Hence the ORTHOGONAL PROJECTION of $\cos 3x$ onto V_0 is 0

(iv) the ORTHOGONAL PROJECTION of $f(x)$ onto V_0 is

$$Pf(x) = \left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle \frac{1}{\sqrt{2\pi}} + \left\langle f, \frac{1}{\sqrt{\pi}} \cos x \right\rangle \frac{1}{\sqrt{\pi}} \cos x + \left\langle f, \frac{1}{\sqrt{\pi}} \sin x \right\rangle \frac{1}{\sqrt{\pi}} \sin x$$

$$\left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x \, dx = 0$$ since INTEGRAND IS ODD

$$\left\langle f, \frac{1}{\sqrt{\pi}} \cos x \right\rangle = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x \cos x \, dx = 0$$ since INTEGRAND IS ODD

$$\left\langle f, \frac{1}{\sqrt{\pi}} \sin x \right\rangle = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x \sin x \, dx = \frac{1}{\sqrt{\pi}} \left[-x \cos x \Big|_{-\pi}^{\pi} + \cancel{\int_{-\pi}^{\pi} \cos x \, dx} \right] =$$

$$= \frac{1}{\sqrt{\pi}} 2\pi = 2\sqrt{\pi}$$

Hence $Pf(x) = 2 \sin x$