Math 4355 – Spring 2017 Name:

TEST #1

No notes or calculators allowed. Please, write clearly and justify all your steps, to get proper credit for your work.

(1)[3 Pts] Let $V = \mathbb{R}^4$ and consider the subspace $V_0 \subset V$ given by

$$V_0 = \text{span}\{(1,0,0,-1),(2,0,0,2)\}.$$

(a) Let V be an inner product space and $V_0 \subset V$. State the definition of the *orthogonal complement* of V_0 in V.

(b) Find the orthogonal complement $V_0^\perp$ of V_0 in V and find an orthonormal basis for $V_0^\perp$.

(2)[3 Pts] Consider the inner product space $V = L^2([0,1])$ with the standard inner product.

(a) Compute the orthogonal projection of the function $f(x) = \sin \pi x$, for $x \in [0,1]$, onto the subspace $V_0 = \text{span}\{\phi, \psi\} \subset L^2[0,1]$, where

$$\phi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) [**2Pts extra credit**] Let $\psi_{2,0}(x) = \psi(2x)$. Plot the function $\psi_{2,0}(x)$, $x \in [0,1]$ and show that $\psi_{2,0}$ is orthogonal to both ϕ and ψ in $L^2[0,1]$.

(3)[4 Pts] Consider the sequence of functions $(f_n) \subset L^2([0,1])$ defined by

$$f_n(x) = \begin{cases} 1 - nx & 0 \le x < \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the graph of $f_n(x)$ for $n = 1, 2, 3$.

(b) Prove that the sequence (f_n) converges to the function $f(x) = 0$, $x \in [0,1]$, in the L^2 norm.

(c) Show that the sequence of numbers (f_n) does not converge pointwise to the functions $f(x) = 0$, $x \in [0,1]$.

① (a) Let V be an I.P. space and $V_0 \subset V$ be a subspace. The orthogonal complement $V_0^\perp = \{v \in V : \langle v, v_0 \rangle = 0 \text{ for all } v_0 \in V_0\}$

(b) Note that $e_1 = (1,0,0,-1)$ and $e_2 = (2,0,0,2)$ are ORTHOGONAL since

$e_1 \cdot e_2 = 2-2=0$

Hence $\{e_1, e_2\}$ is an orthogonal basis of $V_0 = \text{span}\{e_1, e_2\}$

$V_0^\perp = \{v \in V : \langle v, e_1 \rangle = 0 \text{ and } \langle v, e_2 \rangle = 0\}$.

Let $v = (x_1, x_2, x_3, x_4)$

$\langle v, e_1 \rangle = x_1 - x_4 = 0 \quad \Rightarrow \quad x_1 = x_4$

$\langle v, e_2 \rangle = 2x_1 + 2x_4 = 0 \quad \Rightarrow \quad x_1 = -x_4$

$\Rightarrow x_1 = x_4 = 0$

Hence $V_0^\perp = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1 = x_4 = 0\}$

$= \{(0, x_2, x_3, 0) \in \mathbb{R}^4\} = \text{span}\{(0,1,0,0), (0,0,1,0)\}$

This is true since any $x = (0, x_2, x_3, 0)$ can be expressed uniquely as $x_2(0,1,0,0) + x_3(0,0,1,0)$.

The set $\{(0,1,0,0), (0,0,1,0)\}$ is an ONB of $V_0^\perp$

② As shown in class, φ, ψ are orthonormal. Hence they are an ONB of V_0

(a) ORTH. PROJECTION onto V_0 is

$$Pf = \langle f, \varphi \rangle \varphi + \langle f, \psi \rangle \psi$$

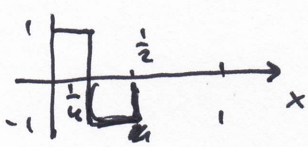
$$\langle \sin \pi x, \varphi(x) \rangle = \int_0^1 \sin \pi x \, dx = -\frac{\cos \pi x}{\pi}\Big|_0^1 = \frac{2}{\pi}$$

$$\langle \sin \pi x, \psi(x) \rangle = \int_0^1 \sin \pi x \, \psi(x) dx = \int_0^{1/2} \sin \pi x \, dx - \int_{\frac{1}{2}}^1 \sin \pi x \, dx$$

$$= \frac{1}{\pi}(-\cos \pi x)\Big|_0^{1/2} + \frac{1}{\pi}\cos \pi x \Big|_{1/2}^1 = \frac{1}{\pi} - 0 - \frac{1}{\pi} + 0 = 0$$

Hence $Pf(x) = \frac{2}{\pi}\varphi(x) = \frac{2}{\pi}\chi_{[0,1]}(x)$

(b) $$\psi_{2,0}(x) = \psi(2x) = \begin{cases} 1 & 0 \le 2x < \frac{1}{2} \\ -1 & 1/2 \le 2x < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & 0 \le x < \frac{1}{4} \\ -1 & \frac{1}{4} \le x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



$$\langle \psi_{2,0}, \varphi \rangle = \int_0^1 \psi(2x) dx = \int_0^{1/4} dx - \int_{\frac{1}{4}}^{1/2} dx = 0$$

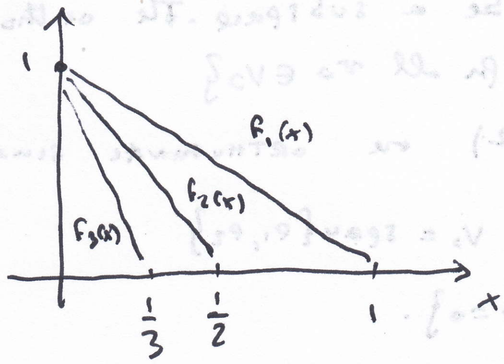
$$\langle \psi_{2,0}, \psi \rangle = \int_0^1 \psi(2x)\psi(x) dx = \int_0^{1/2} \psi(2x) dx = 0$$

↑ since $\psi_{2,0}(x) = 0$ if $x \in [1/2, 1]$

since $\psi(x) = 1$ if $x \in [0, \frac{1}{2}]$

3

(a)



(b) $$\|f_n - f\|^2_{L^2} = \int_0^1 |f_n(x)|^2\,dx = \int_0^{1/n} (1-nx)^2\,dx =$$

let $f = 0$

$$= \int_0^{1/n} (1 - 2nx + n^2x^2)\,dx = \left(x - nx^2 + n^2\frac{x^3}{3}\right)\Big|_0^{1/n}$$

$$= \frac{1}{n} - \frac{1}{n} + \frac{1}{3n} = \frac{1}{3n}$$

Thus $\lim_{n\to\infty} \|f_n - f\|_{L^2} = \lim_{n\to\infty} \frac{1}{\sqrt{3n}} = 0$

(c) $f_n(0) = 1$ for any $n \in \mathbb{N}$

Hence $\lim_{n\to\infty} f_n(0) = 1$ and $f_n(x)$ does NOT converge pointwise to the function $f(x) = 0$ $\forall x \in [0,1]$.