Math 4355 – Spring 2021 Name:

TEST #1 - version 1

Please, show your work, justify every step and write legibly. When you are done, scan, save the file as LASTNAME_FIRSTNAME_T1.pdf and email to `dlabate@math.uh.edu` or `dlabate@uh.edu`. NOTE: You need to send your email before 11:30AM on Feb 25 to receive credit.

(1)[4 Pts]

(a) Let $V = \mathbb{R}^3$ and consider the subspace $V_0 \subset V$ given by

$$V_0 = \text{span}\,\{(1,-1,1)\}.$$

Find the orthogonal complement $V_0^\perp$ of V_0 in V and an orthonormal basis for this space.

(b) Let $V = \mathbb{R}^4$ and consider the subspace $V_0 \subset V$ given by

$$V_0 = \text{span}\,\{(1,-1,1,0)\}.$$

Find the orthogonal complement $V_0^\perp$ of V_0 in V and an orthonormal basis for this space.

Solution

(a) The orthogonal complement if V_0 is the plane of equation

$$x - y + z = 0$$

To find a ONB of $V_0^\perp$, take $v_1 = (1,1,0) \in V_0^\perp$ (it is easy to see that v_1 lies in the plane) and compute the cross product of v_1 and $(1,-1,1,0)$. We find

$$v_2 = (1,-1,1)\langle(1,1,0) = (-1,1,2)$$

Hence an ONB of $V_0^\perp$ is $\{e_1, e_2\}$ where

$$e_1 = \frac{1}{\sqrt{2}}(1,1,0), \quad e_1 = \frac{1}{\sqrt{6}}(-1,1,2)$$

(b) It is sufficient to find 3 vectors orthogonal to $(1,-1,1,0)$. Due to its special form and the result from part (a) we can choose

$$e_1 = \frac{1}{\sqrt{2}}(1,1,0,0), \quad e_1 = \frac{1}{\sqrt{6}}(-1,1,2,0) \quad e_2 = (0,0,0,1)$$

We have that $V_0^\perp = \text{span}\,\{(e_1, e_2, e_3)\}$ and $\{(e_1, e_2, e_3)\}$ is an ONB of this space.

(2)[4 Pts] Consider the inner product space $V = L^2([-\pi, \pi])$ with the standard inner product. Compute the orthogonal projection of the function

$$f(x) = \begin{cases} -1 & \text{if } -\pi \le x < 0 \\ \ \ 1 & \text{if } \quad 0 \le x \le \pi \end{cases}$$

onto the subspace $V_0 = \text{span}\{1, \sin x, \cos x, \sin 2x, \cos 2x\} \subset L^2([-\pi, \pi])]$.

Solution

An ONB of V_0 is given by the vectors

$$e_1(x) = \tfrac{1}{\sqrt{2\pi}}, e_2(x) = \tfrac{1}{\sqrt{\pi}} \sin x, e_3(x) = \tfrac{1}{\sqrt{\pi}} \cos x, e_4(x) = \tfrac{1}{\sqrt{\pi}} \sin 2x, e_5(x) = \tfrac{1}{\sqrt{\pi}} \cos 2x$$

Since f is odd and the functions 1, $\cos x$, $\cos 2x$ are even, then $\langle f, 1\rangle = \langle f, \cos x\rangle = \langle f, \cos 2x\rangle = 0$

$$\langle f, \sin x\rangle = 2\int_0^\pi \sin x \, dx = -2\cos x|_0^\pi = 4$$

$$\langle f, \sin 2x\rangle = 2\int_0^\pi \sin 2x \, dx = -\cos 2x|_0^\pi = 0$$

It follows that the orthogonal projection of f onto V_0 is

$$f_0(x) = \sum_{k=1}^{5} \langle f, e_k\rangle e_k(x) = \langle f, e_2\rangle e_2(x) = \frac{4}{\pi}\sin x$$

(3)[5 Pts] Consider the sequence of functions $(f_n) \subset L^2([0,1])$ defined by

$$f_n(x) = \begin{cases} 1-(nx)^2 & 0 \le x < \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the graph of $f_n(x)$ for $n = 1, 2, 3$.

(b) Prove that the sequence (f_n) converges to the function $f(x) = 0$, $x \in [0,1]$, in the L^2 norm.

(c) Show that the sequence $(f_n(x))$ does not converge pointwise to the functions $f(x) = 0$, $x \in [0,1]$ [Hint: consider the point $x = 0$]

Solution

(a)

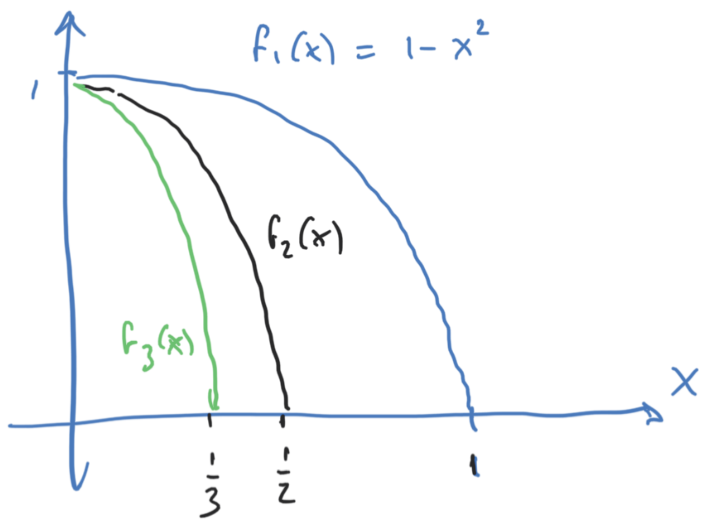

Figure 1: Sketch of $f_n(x)$, for $n = 1, 2, 3$

(b) We will show that

$$\lim_{n\to\infty} \|f_n - 0\|^2 = \lim_{n\to\infty} \|f_n\|^2 = 0$$

By direct calculation

$$\|f_n\|^2 = \int_0^{\frac{1}{n}} (1-(nx)^2)^2 = \int_0^{\frac{1}{n}} \left(1 - 2(nx)^2 + (nx)^4\right) dx = (x - \frac{2}{3}n^2x^3 + \frac{1}{5}n^4x^5)\Big|_0^{\frac{1}{n}}$$

Hence

$$\|f_n\|^2 = \frac{1}{n} - \frac{2}{3}\frac{1}{n} + \frac{1}{5}\frac{1}{n} = \frac{8}{15}\frac{1}{n} \to 0 \text{ as } n \to \infty$$

This shows that (f_n) converges to 0 in norm.

(c) $(f_n(x))$ does not converge pointwise to the functions $f(x) = 0$. In fact, for $x = 0$, $f_n(0) = 1$ for all n, hence $\lim_{n\to\infty} f_n(0) = 1$.