

TEST #1 - version 2

Please, show your work, justify every step and write legibly. When you are done, scan, save the file as LASTNAME_FIRSTNAME_T1.pdf and email to dlabate@math.uh.edu or dlabate@uh.edu.

NOTE: You need to send your email before 11:30AM on Feb 25 to receive credit.

(1)[4 Pts]

(a) Let $V = \mathbb{R}^3$ and consider the subspace $V_0 \subset V$ given by

$$V_0 = \text{span}\{(-1, 1, -1)\}.$$

Find the orthogonal complement V_0^\perp of V_0 in V and an orthonormal basis for this space.

(b) Let $V = \mathbb{R}^4$ and consider the subspace $V_0 \subset V$ given by

$$V_0 = \text{span}\{(-1, 1, -1, 0)\}.$$

Find the orthogonal complement V_0^\perp of V_0 in V and an orthonormal basis for this space.

Solution

(a) The orthogonal complement of V_0 is the plane of equation

$$x - y + z = 0$$

To find a ONB of V_0^\perp , take $v_1 = (1, 1, 0) \in V_0^\perp$ (it is easy to see that v_1 lies in the plane) and compute the cross product of v_1 and $(1, -1, 1, 0)$. We find

$$v_2 = (1, -1, 1) \times (1, 1, 0) = (-1, 1, 2)$$

Hence an ONB of V_0^\perp is $\{e_1, e_2\}$ where

$$e_1 = \frac{1}{\sqrt{2}}(1, 1, 0), \quad e_2 = \frac{1}{\sqrt{6}}(-1, 1, 2)$$

(b) It is sufficient to find 3 vectors orthogonal to $(-1, 1, -1, 0)$. Due to its special form and the result from part (a) we can choose

$$e_1 = \frac{1}{\sqrt{2}}(1, 1, 0, 0), \quad e_2 = \frac{1}{\sqrt{6}}(-1, 1, 2, 0) \quad e_3 = (0, 0, 0, 1)$$

We have that $V_0^\perp = \text{span}\{(e_1, e_2, e_3)\}$ and $\{(e_1, e_2, e_3)\}$ is an ONB of this space.

(2)[4 Pts] Consider the inner product space $V = L^2([-π, π])$ with the standard inner product. Compute the orthogonal projection of the function

$$f(x) = \begin{cases} 1 & \text{if } -\pi \leq x < 0 \\ -1 & \text{if } 0 \leq x \leq \pi \end{cases}$$

onto the subspace $V_0 = \text{span}\{1, \sin x, \cos x, \sin 2x, \cos 2x\} \subset L^2([-π, π])$.

Solution

An ONB of V_0 is given by the vectors

$$e_1(x) = \frac{1}{\sqrt{2\pi}}, e_2(x) = \frac{1}{\sqrt{\pi}} \sin x, e_3(x) = \frac{1}{\sqrt{\pi}} \cos x, e_4(x) = \frac{1}{\sqrt{\pi}} \sin 2x, e_5(x) = \frac{1}{\sqrt{\pi}} \cos 2x$$

Since f is odd and the functions $1, \cos x, \cos 2x$ are even, then $\langle f, 1 \rangle = \langle f, \cos x \rangle = \langle f, \cos 2x \rangle = 0$

$$\langle f, \sin x \rangle = -2 \int_0^\pi \sin x \, dx = 2 \cos x \Big|_0^\pi = -4$$

$$\langle f, \sin 2x \rangle = -2 \int_0^\pi \sin 2x \, dx = \cos 2x \Big|_0^\pi = 0$$

It follows that the orthogonal projection of f onto V_0 is

$$f_0(x) = \sum_{k=1}^5 \langle f, e_k \rangle e_k(x) = \langle f, e_2 \rangle e_2(x) = -\frac{4}{\pi} \sin x$$

(3)[5 Pts] Consider the sequence of functions $(f_n) \subset L^2([0, 1])$ defined by

$$f_n(x) = \begin{cases} 1 - (nx)^2 & 0 \leq x < \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Sketch the graph of $f_n(x)$ for $n = 1, 2, 3$.
 (b) Prove that the sequence (f_n) converges to the function $f(x) = 0$, $x \in [0, 1]$, in the L^2 norm.
 (c) Show that the sequence $(f_n(x))$ does not converge pointwise to the functions $f(x) = 0$, $x \in [0, 1]$ [Hint: consider the point $x = 0$]

Solution

(a)

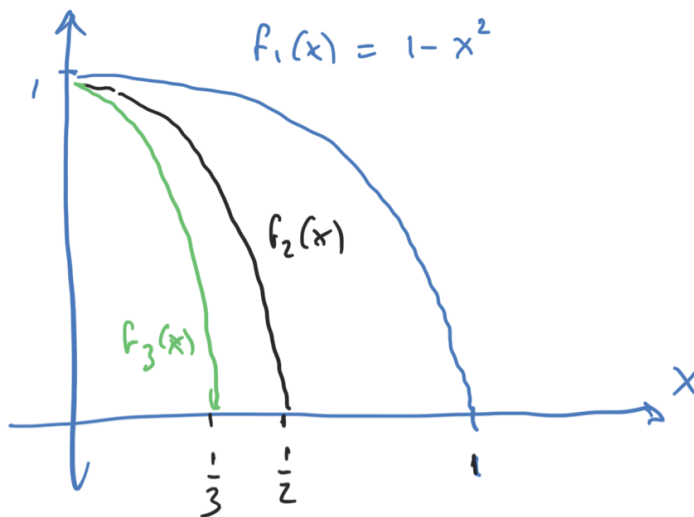


Figure 1: Sketch of $f_n(x)$, for $n = 1, 2, 3$

(b) We will show that

$$\lim_{n \rightarrow \infty} \|f_n - 0\|^2 = \lim_{n \rightarrow \infty} \|f_n\|^2 = 0$$

By direct calculation

$$\|f_n\|^2 = \int_0^{\frac{1}{n}} (1 - (nx)^2)^2 dx = \int_0^{\frac{1}{n}} (1 - 2(nx)^2 + (nx)^4) dx = \left(x - \frac{2}{3}n^2x^3 + \frac{1}{5}n^4x^5 \right) \Big|_0^{\frac{1}{n}}$$

Hence

$$\|f_n\|^2 = \frac{1}{n} - \frac{2}{3} \frac{1}{n} + \frac{1}{5} \frac{1}{n} = \frac{8}{15} \frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

This shows that (f_n) converges to 0 in norm.

(c) $(f_n(x))$ does not converge pointwise to the functions $f(x) = 0$. In fact, for $x = 0$, $f_n(0) = 1$ for all n , hence $\lim_{n \rightarrow \infty} f_n(0) = 1$.