Math 4355 – Spring 2017 Name:

TEST #2

NOTE: *You can use 1 double-sided page of notes. No calculators allowed. Please, write clearly and justify all your steps to get proper credit for your work.*

- (1) [6 Pts] Without computing the Fourier coefficients, answer the following questions about the convergence of the following series. Justify your answer.

(a) Does the sine series of $f(x) = \sin x$, on the interval $0 \le x \le \pi$, converge uniformly to f?

(b) Does the cosine series of $f(x) = \sin x$, on the interval $0 \le x \le \pi$, converge uniformly to f?

- (2) [6 Pts]

(a) Expand the function $f(x) = x^2$ in a Fourier sine series valid in the interval $0 \le x \le 1$.

(b) Does the Fourier sine series converge to f uniformly on $[-1 \le x \le 1]$? Justify.

- (3) [6 Pts] Let

$$\phi(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise;} \end{cases} \qquad g(x) = \begin{cases} x & 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $h(x) = (\phi * g)(x)$.

(b) Sketch the graph of ϕ, g, h over the interval $[-2, 4]$.

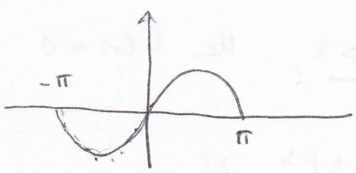
- (4) [6 Pts] Let $f(x) = e^{-a|x|}$, where $a > 0$.

(a) Compute the Fourier transform of $f(x)$.

(b) Let $F_a(u) = \frac{2a}{a^2+u^2}$. Show that $F_a * F_b = 2\pi\, F_{a+b}$
(HINT: use part (a) and then the 'convolution' theorem).

TEST #2 - SOLUTION

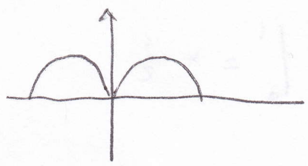
(1) (a) $f(x) = \sin x$ Sine series



The 2π-periodization is continuous

Thus the sine series converges UNIFORMLY to f

(b) $f(x) = \sin x$ Cosine series



The 2π-periodization is continuous.

Thus the cosine series converges uniformly to f

(2) $f(x) = x^2$

(a) Fourier sine series is $F(x) = \sum_{k=1}^{\infty} b_k \sin k\pi x$ $\qquad b_k = 2\int_0^1 x^2 \sin(k\pi x)\,dx$

$$b_k = 2\int_0^1 x^2 \sin k\pi x\,dx = 2\left[-\frac{x^2}{k\pi}\cos k\pi x\Big|_0^1 + \frac{2}{k\pi}\int_0^1 x\cos k\pi x\,dx\right]$$

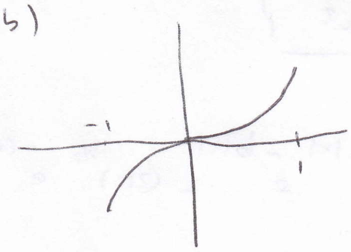
$$= 2\left[-\frac{x^2}{k\pi}\cos k\pi x\Big|_0^1 + \cancel{\frac{2}{(k\pi)^2}x\sin k\pi x}\Big|_0^1 - \frac{2}{(k\pi)^2}\int_0^1 \sin k\pi x\,dx\right]$$

$$= 2\left[-\frac{x^2}{k\pi}\cos k\pi x\Big|_0^1 + \frac{2}{(k\pi)^3}\cos k\pi x\Big|_0^1\right]$$

$$= 2\left[-\frac{1}{k\pi}\cos k\pi + \frac{2}{(k\pi)^3}\cos k\pi - \frac{2}{(k\pi)^3}\right]$$

$$= 2\left(-\frac{(-1)^k}{k\pi} + \frac{2}{(k\pi)^3}\left[(-1)^k - 1\right]\right)$$

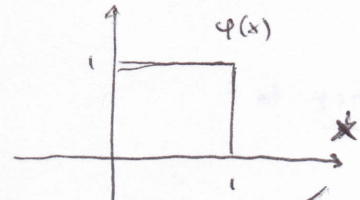
(b)



The 2π-periodization is discontinuous at $x = \pm 1$

Hence the SINE SERIES of $f(x) = x^2$ does NOT converge UNIFORMLY to f

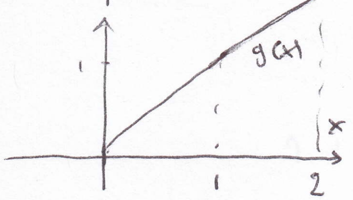
(3) $h(x) = \varphi * g(x) = \int_{\mathbb{R}} \varphi(t)\, g(x-t)\, dt = \int_0^1 g(x-t)\, dt$



if $x<0$ or $x-2>1 \iff x>3$, then $h(x)=0$

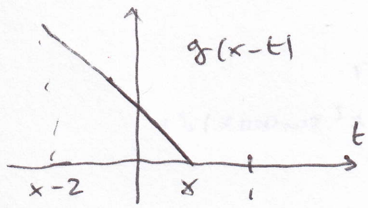
if $0\le x \le 1$ then

$$h(x) = \int_0^x (x-t)\,dt = xt - \frac{t^2}{2}\Big|_0^x = \frac{x^2}{2}$$



if $1 \le x \le 2$

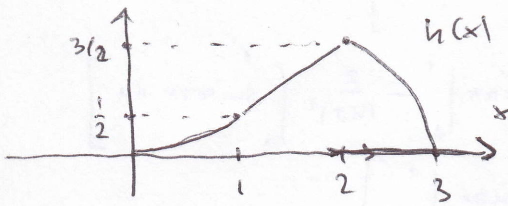
$$h(x) = \int_0^1 (x-t)\,dt = xt - \frac{t^2}{2}\Big|_0^1 = x - \frac{1}{2}$$



if $2 \le x \le 3$,

$$h(x) = \int_{x-2}^1 (x-t)\,dt = xt - \frac{t^2}{2}\Big|_{x-2}^1 = -\frac{x^2}{2} + x + \frac{3}{2} =$$

$$= -\frac{1}{2}(x-3)(x+1)$$



(4) (a) $\mathcal{F}\left[e^{-a|x|}\right](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{ax - i\omega x}\, dx + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ax - i\omega x}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ax - i\omega x}}{a - i\omega}\Big|_{-\infty}^0 - \frac{e^{-ax - i\omega x}}{a + i\omega}\Big|_0^\infty \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a - i\omega} + \frac{1}{a + i\omega} \right) = \boxed{\frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + \omega^2}}$$

(b) By part (a) $\mathcal{F}^{-1}(F_a)(x) = \sqrt{2\pi}\, e^{-a|x|}$

$$\mathcal{F}^{-1}[F_a * F_b] = \sqrt{2\pi}\, \mathcal{F}^{-1}(F_a)(x)\, \mathcal{F}^{-1}(F_b)(x) = \sqrt{2\pi}\, 2\pi\, e^{-a|x|}\, e^{-b|x|} = (2\pi)^{3/2} e^{-(a+b)|x|}$$

$$= 2\pi\, \mathcal{F}^{-1}[F_{a+b}].$$

Thus $F_a * F_b = 2\pi F_{a+b}$