**Math 4355 – Spring 2021** Name:

# TEST #2 - version 1

Please, show your work, justify every step and write legibly. When you are done, scan, save the file as LASTNAME_FIRSTNAME_T2.pdf and email to `dlabate@math.uh.edu` or `dlabate@uh.edu`. NOTE: You need to send your email before 11:30AM on Feb 25 to receive credit.

- (1) [8 Pts]

(a) Compute the sine series of $f(x) = \sin x$, on the interval $0 \le x \le \pi$.
(b) Does the sine series of $f$ converge uniformly to $f$ in $[0, \pi]$? If not, where does it fail to converge? Justify.
(c) Without computing the Fourier coefficients, set up the computation of the cosine series of $f(x) = \sin x$, on the interval $0 \le x \le \pi$.
(d) Does the cosine series of $f$ converge uniformly to $f$ in $[0, \pi]$? If not, where does it fail to converge? Justify.

*(a)* $F(x) = \sum_{k=1}^{\infty} b_k \sin kx$
*where*
$b_k = \frac{2}{\pi}\int_0^{\pi} \sin x \, \sin kx \, dx = 1$, *if* $k = 1$ *and* $b_k = 0$ *if* $k > 1$ *due to the orthogonality properties of the set* $\sin kx$.

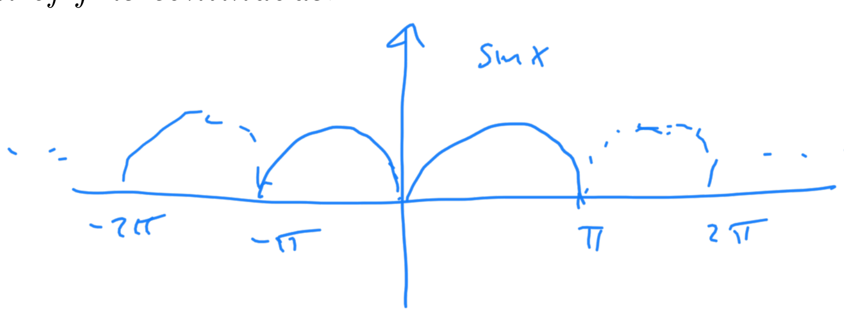
*Thus the sine series of* $f$ *is* $F(x) = \sin x$

*Note: there is no need to integrate as* $f$ *is already in the form of a sine series.*

*(b) The sine series of* $f$ *converge uniformly to* $f$ *in* $[0, \pi]$ *since the* $2\pi$ *periodization of* $f$ *is continuous.*

*(c)* $F(x) = a_o + \sum_{k=1}^{\infty} a_k \cos kx$ *where*
$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx$
*and*
$a_k = \frac{2}{\pi}\int_0^{\pi} \sin x \, \cos kx \, dx$

*(d) The cosine series of* $f$ *converge uniformly to* $f$ *in* $[0, \pi]$ *since the* $2\pi$ *periodization of* $f$ *is continuous.*

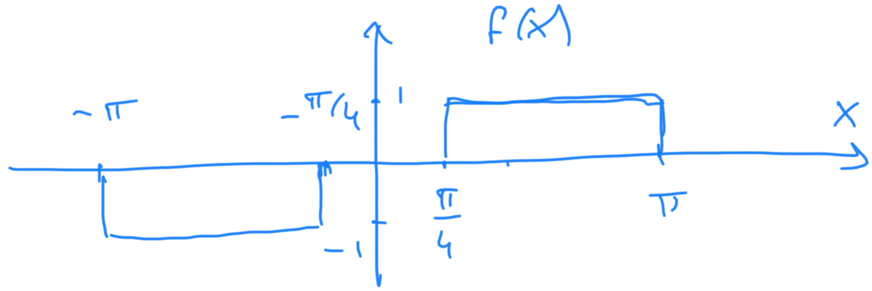


• (2) [8 Pts] Consider the function

$$f(x) = \begin{cases} -1 & \text{if } -\pi \le x \le -\frac{\pi}{4} \\ 0 & \text{if } -\frac{\pi}{4} < x < \frac{\pi}{4} \\ 1 & \text{if } \frac{\pi}{4} \le x \le \pi \end{cases}$$

(a) Sketch the function.
(b) Compute the Fourier series of $f$, valid in the interval $[-\pi, \pi]$. (Hint: Take advantage of even/odd properties to reduce calculations.)
(c) Does the Fourier series converge to $f$ uniformly on $[-\pi \le x \le \pi]$? If not, where does it fail to converge? Justify.

(a)




*(b) The function $f$ is odd. Thus*

$$F(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

*where*

$$b_k = \frac{2}{\pi} \int_{\frac{\pi}{4}}^{\pi} \sin kx \, dx = \frac{2}{k\pi} (\cos kx) \Big|_{\frac{\pi}{4}}^{\pi} = \frac{2}{k\pi} \left( \cos k\frac{\pi}{4} - \cos k\pi \right)$$

*(c) In the interval $[-\pi, \pi]$, the $2\pi$ periodization of $f$ is discontinuous at $x = \pm\pi, \pm\frac{\pi}{4}$. Hence $F$ does not converge uniformly to $f$. These 4 points of discontinuity are the points where $F$ fails to converge to $f$.*

- (3) [4 Pts]

(a) Compute the Fourier series of $f(x) = \sin^2 x$ valid in the interval $[-\pi, \pi]$

(b) Compute the Complex Fourier series of $f(x) = \sin^2 x$ valid in the interval $[-\pi, \pi]$ [Hint: there is no need of integration. You can derive it using the result of part (a).]

*(a) Using the identity:* $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, *it follows that the Fourier series of f is*

$$F(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$$

*(b) Using Euler's formula* $e^{iy} = \cos y + i \sin y$, *we obtain that* $\cos y = \frac{e^{iy}+e^{-iy}}{2}$. *Hence, from part (a) we have that*

$$F(x) = \frac{1}{2} - \frac{1}{2}\left(\frac{e^{i2x} + e^{-i2x}}{2}\right) = \frac{1}{2} - \frac{1}{4}e^{i2x} - \frac{1}{4}e^{-i2x}$$