Math 4355 – Spring 2021 Name: SOLUTION

TEST #2 - version 2

Please, show your work, justify every step and write legibly. When you are done, scan, save the file as LASTNAME_FIRSTNAME_T2.pdf and email to dlabate@math.uh.edu or dlabate@uh.edu. NOTE: You need to send your email before 11:30AM on Feb 25 to receive credit.

- (1) [8 Pts]

(a) Compute the sine series of $f(x) = \sin x$, on the interval $0 \leq x \leq \pi$.

(b) Does the sine series of f converge uniformly to f in $[0, \pi]$? If not, where does it fail to converge? Justify.

(c) Without computing the Fourier coefficients, set up the computation of the cosine series of $f(x) = \sin x$, on the interval $0 \leq x \leq \pi$.

(d) Does the cosine series of f converge uniformly to f in $[0, \pi]$? If not, where does it fail to converge? Justify.

(a) $F(x) = \sum_{k=1}^{\infty} b_k \sin kx$
where
$b_k = \frac{2}{\pi} \int_0^{\pi} \sin x \, \sin kx \, dx = 1$, *if* $k = 1$ *and* $b_k = 0$ *if* $k > 1$ *due to the orthogonality properties of the set* $\sin kx$.

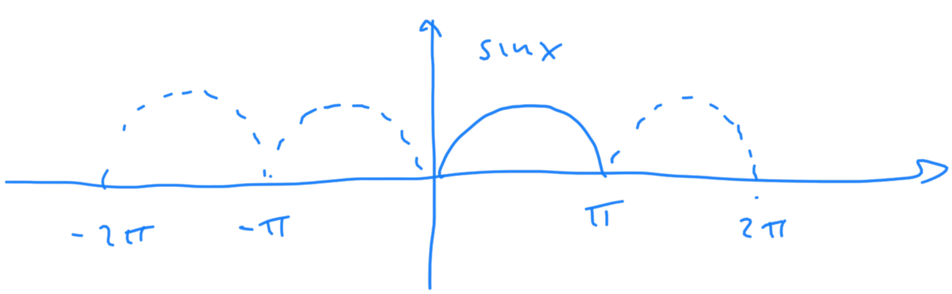
Thus the sine series of f *is* $F(x) = \sin x$

Note: there is no need to integrate as f *is already in the form of a sine series.*

(b) The sine series of f *converge uniformly to* f *in* $[0, \pi]$ *since the* 2π *periodization of* f *is continuous.*

(c) $F(x) = a_o + \sum_{k=1}^{\infty} a_k \cos kx$ *where*
$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx$
and
$a_k = \frac{2}{\pi} \int_0^{\pi} \sin x \, \cos kx \, dx$

(d) The cosine series of f *converge uniformly to* f *in* $[0, \pi]$ *since the* 2π *periodization of* f *is continuous.*



- (2) [8 Pts] Consider the function

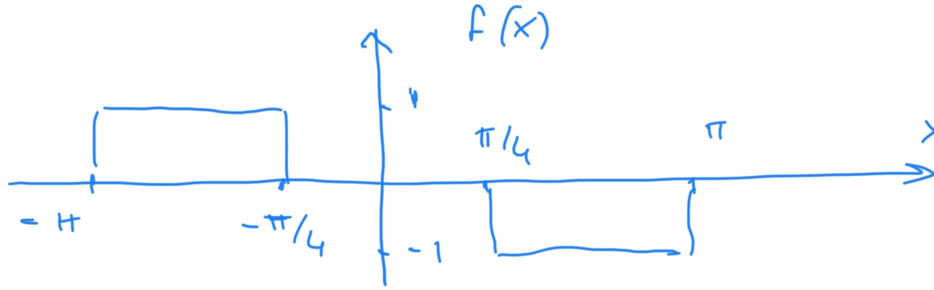
$$f(x) = \begin{cases} 1 & \text{if } -\pi \le x \le -\frac{\pi}{4} \\ 0 & \text{if } -\frac{\pi}{4} < x < \frac{\pi}{4} \\ -1 & \text{if } \quad \frac{\pi}{4} \le x \le \pi \end{cases}$$

(a) Sketch the function.

(b) Compute the Fourier series of f, valid in the interval $[-\pi, \pi]$. (Hint: Take advantage of even/odd properties to reduce calculations.)

(c) Does the Fourier series converge to f uniformly on $[-\pi \le x \le \pi]$? If not, where does it fail to converge? Justify.

(a)



(b) The function f is odd. Thus

$$F(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

where

$$b_k = -\frac{2}{\pi}\int_{\frac{\pi}{4}}^{\pi} \sin kx\, dx = -\frac{2}{k\pi}(\cos kx)\Big|_{\frac{\pi}{4}}^{\pi} = \frac{2}{k\pi}\left(\cos k\pi - \cos k\frac{\pi}{4}\right)$$

(c) In the interval $[-\pi, \pi]$, the 2π periodization of f is discontinuous at $x = \pm\pi, \pm\frac{\pi}{4}$. Hence F does not converge uniformly to f. These 4 points of discontinuity are the points where F fails to converge to f.

- (3) [4 Pts]

(a) Compute the Fourier series of $f(x) = \sin^2 x$ valid in the interval $[-\pi, \pi]$

(b) Compute the Complex Fourier series of $f(x) = \sin^2 x$ valid in the interval $[-\pi, \pi]$ [Hint: there is no need of integration. You can derive it using the result of part (a).]

(a) Using the identity: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$*, it follows that the Fourier series of f is*

$$F(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$$

(b) Using Euler's formula $e^{iy} = \cos y + i \sin y$*, we obtain that* $\cos y = \frac{e^{iy}+e^{-iy}}{2}$*. Hence, from part (a) we have that*

$$F(x) = \frac{1}{2} - \frac{1}{2}(\frac{e^{i2x} + e^{-i2x}}{2}) = \frac{1}{2} - \frac{1}{4}e^{i2x} - \frac{1}{4}e^{-i2x}$$