Math 4355 – Spring 2017 Name:

TEST #3

Only one double-sided page of notes is allowed. Please, write clearly and justify all your steps to get proper credit for your work.

- (1) [3 Pts] Let ψ be the Haar wavelet.

(a) Compute explicitly the support of $\psi_{j,k}(x) = \psi(2^j x - k)$, $j, k \in \mathbb{Z}$. Recall that, for a function f, the support is the set $\mathrm{supp} f = \{x : f(x) \neq 0\}$.

(b) Prove that the set $\{\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ is orthogonal. Hint: Use part (a) to derive that $\langle \psi_{j,k}, \psi_{j,k'} \rangle = 0$, if $k \neq k'$. Also show that $\langle \psi_{j,k}, \psi_{j',k'} \rangle = 0$ if $j \neq j'$.

- (2) [8 Pts] Let ϕ and ψ be the Haar scaling and wavelet functions respectively. Let V_j and W_j be the spaces generated by $\phi(2^j x - k)$, $k \in \mathbb{Z}$ and $\psi(2^j x - k)$, $k \in \mathbb{Z}$, respectively. Consider the function defined on $0 \leq x < 1$ given by

$$f(x) = \begin{cases} -1 & 0 \leq x < 1/8 \\ 0 & 1/8 \leq x < 2/8 \\ 0 & 2/8 \leq x < 3/8 \\ 1 & 3/8 \leq x < 4/8 \\ 1 & 4/8 \leq x < 5/8 \\ 2 & 5/8 \leq x < 6/8 \\ 3 & 6/8 \leq x < 7/8 \\ 5 & 7/8 \leq x < 1. \end{cases}$$

(a) Express f in terms of the basis for V_3.

(b) Find the Haar wavelet decomposition of f. That is, decompose f into its component parts in V_0, W_0, W_1 and W_2.

(c) Sketch each of the component parts of the Haar decomposition of f.

(d) On the interval $0 \leq x < 1$, what is the dimension of the space V_2? what is the dimension of W_2? Justify your answer.

- (3)[3 Pts] Reconstruct $f \in V_3$ over the interval $0 \leq x < 1$, given the coefficients in its Haar wavelet decomposition:

$$a^2 = (0, -1/2, 5/2, -3/2), \quad b^2 = (1, -5/2, -1/2, 1/2).$$

Notice that a^2 are the coefficients associated with the approximation space V_2 and b^2 are the coefficients associated with the wavelet space W_2.

Next, sketch the function f.

- (4)[3 Pts] Suppose that ϕ is the scaling function of a multiresolution analysis and that it has compact support.

(a) Prove that in this case there are only a finite number of nonzero coefficients p_k in the scaling relation

$$\phi(x) = \sum_{k \in \mathbb{Z}} p_k \, \phi(2x - k), \quad p_k = 2 \int_{\mathbb{R}} \phi(x) \, \overline{\phi(2x - k)} \, dx.$$

(b) What can you say about the support of the wavelet ψ? Justify your answer. (Hint: recall that $\psi(x) = \sum_{k \in \mathbb{Z}} (-1)^k \, \overline{p_{1-k}} \, \phi(2x - k)$.)

TEST #3 SOLUTION

(1) (a) $\text{supp}\,\varphi = [0,1]$

$2^j x - k \in [0,1] \iff 2^j x \in [k, k+1] \iff x \in [2^{-j}k, 2^{-j}(k+1)]$

Hence $\boxed{\text{supp}\,\varphi_{j,k} = [2^{-j}k, 2^{-j}(k+1)]} := I_{j,k}$

(b) Since $|I_{j,k}| = 2^{-j}$ and its endpoints are of form $2^{-j}k$, $I_{j,k} \cap I_{j,k'} = \phi$, for $k \neq k'$.

Thus $\langle \varphi_{j,k}, \varphi_{j,k'} \rangle = 0$, if $k \neq k'$

If $j > j'$, then $2^{-j} < 2^{-j'}$, hence either $I_{j,k} \cap I_{j',m} = \phi$ or $I_{j,k} \subset I_{j',m}$ and the smaller interval is entirely contained in larger one since endpoints are of form $2^{-j}k$, the overlap can only happen when $I_{j',m}$ is constant. Hence it must be $\langle \varphi_{j,k}, \varphi_{j',m} \rangle = 0$, if $j \neq j'$.

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(2) (a) $f(x) = -\varphi(8x) + \varphi(8x-3) + \varphi(8x-4) + 2\varphi(8x-5) + 3\varphi(8x-6) + 5\varphi(8x-7)$

(b)

a_3 : -1 0 0 1 1 2 3 5

a_2 : $-1/2$ $1/2$ $3/2$ 4

[b_2 : $-1/2$ $-1/2$ $-1/2$ -1

a_1 : 0 $11/4$

[b_1 : $-1/2$ $-5/4$

[a_0 : $11/8$

[b_0 : $-11/8$

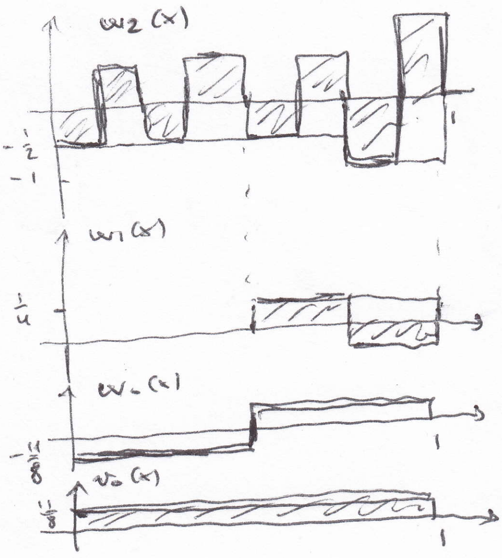
$f(x) = -\frac{1}{2}\psi(4x) - \frac{1}{2}\psi(4x-1) - \frac{1}{2}\psi(4x-2) - \psi(4x-3) +$ $\in W_2$

$- \frac{1}{2}\psi(2x) - \frac{5}{4}\psi(2x-1)$ $\in W_1$

$- \frac{11}{8}\psi(x)$ $\in W_0$

$+ \frac{11}{8}\varphi(x)$ $\in V_0$

(c)



(d) $V_2 = \text{span}\{\varphi(4x-k) : k \in \mathbb{Z}, \text{ with support in } [0,1].\}$

$= \text{span}\{\varphi(4x-k) : 0 \le k \le 3\}$

$\boxed{\dim(V_2) = 4}$

Similarly,

$W_2 = \text{span}\{\psi(4x-k) : k \in \mathbb{Z} \text{ w/ support in } [0,1]\}$

$= \text{span}\{\psi(4x-k) : 0 \le k \le 3\}$

$\boxed{\dim W_2 = 4}$

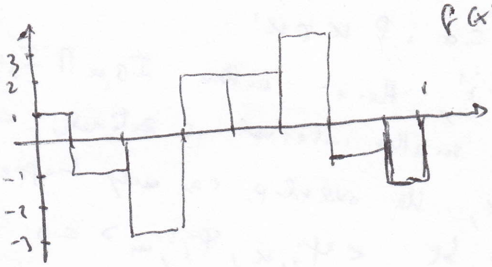
(3)

$a_2 = (0, -1/2, 5/2, -3/2)$ $\qquad$ $b_2 = (1, -5/2, -1/2, 1/2)$

$$a_3(n) = \begin{cases} a_2(k) + b_2(k) & \text{if } n = 2k \\ a_2(k) - b_2(k) & \text{if } n = 2k+1 \end{cases}$$

$$a_3 = (1, -1, -3, 2, 2, 3, -1, -2)$$

$f(x) \in V_3$



(4) (a) If $\varphi(x)$ has compact support, then $\mathrm{supp}(\varphi)$ and $\mathrm{supp}(\varphi(2\cdot - k))$ are disjoint for k sufficiently large.

In fact, $\mathrm{supp}(\varphi) = I$, $\mathrm{supp}(\varphi(2\cdot - k)) = \frac{1}{2} T_k I$ for $|k|$ large enough

It follows that $p_k = 2\int \varphi(x)\overline{\varphi(2x-k)}\,dx = 0$

(b) Since only finitely many p_k are nonzero and φ has compact support, then

$$\psi(x) = \sum_{\text{finite } k} (-1)^k \overline{p_{1-k}}\, \varphi(2x-k)$$

has also compact support.