Math 4355 – Spring 2021 Name: SOLUTION

TEST #3 - version 1

Please, show your work, justify every step and write legibly. When you are done, scan, save the file as LASTNAME_FIRSTNAME_T3.pdf and email to `dlabate@math.uh.edu` or `dlabate@uh.edu`. NOTE: You need to send your email before 11:30AM on April 20 to receive credit.

- (1) [3 Pts] Let $A > 0$ be a fixed number. Does the following functions define a causal filter? Justify your answer.

(a) $h_1(t) = \begin{cases} e^{At} & \text{if } t < 0; \\ e^{-At} & \text{if } t \geq 0. \end{cases}$

(b) $h_2(t) = \begin{cases} 0 & \text{if } t < 0; \\ e^{-At} & \text{if } t \geq 0. \end{cases}$

SOLUTION: A causal filter h has the property that $h(t) = 0$ of $t < 0$. Thus h_1 is not causal, whereas h_2 is causal.

- (2) [9 Pts] Let $f(t) = \begin{cases} 1 & \text{if } -\pi \leq t \leq \pi; \\ 0 & \text{otherwise.} \end{cases}$

We found that its Fourier transform is $\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin(\pi\omega)}{\omega}$.

(a) Use the Fourier transform of f and the properties of the Fourier transform to compute the Fourier transform $\hat{c}(\omega)$ of the function $c(t) = (f * f)(t)$

(b) Use the Fourier transform of f and the properties of the Fourier transform to compute the Fourier transform of

$$h(t) = \begin{cases} 1 & \text{if } -1 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) Use the Fourier transform of h and the properties of the Fourier transform to compute the Fourier transform of

$$g(t) = \begin{cases} t^2 & \text{if } -1 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

SOLUTION:
(a) By the Fourier convolution theorem,

$$\hat{c}(\omega) = \sqrt{2\pi}\hat{f}(\omega)\hat{f}(\omega) = 2\sqrt{\frac{2}{\pi}} \frac{\sin^2(\pi\omega)}{\omega^2}$$

(b) Since $h(t) = f(\pi t)$, then

$$\hat{h}(\omega) = \frac{1}{\pi}\hat{f}(\frac{\omega}{\pi}) = \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}$$

(c) By the multiplication property of the Fourier transform,

$$\hat{g}(\omega) = \mathcal{F}[t^2 fh(t)](\omega) = i^2 \frac{d^2}{d\omega^2}\hat{h}(\omega) = -\frac{d^2}{d\omega^2}\left(\sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}\right)$$

$$\frac{d^2}{d\omega^2}\left(\frac{\sin\omega}{\omega}\right) = \frac{d}{d\omega}\left(\frac{\omega\cos\omega - \sin\omega}{\omega^2}\right) = \frac{d}{d\omega}\left(\frac{\cos\omega}{\omega} - \frac{\sin\omega}{\omega^2}\right)$$

$$= \frac{-\omega\sin\omega - \cos\omega}{\omega^2} - \frac{\omega^2\cos\omega - 2\omega\sin\omega}{\omega^4} = \frac{-2\omega^2\cos\omega - (\omega^3 - 2\omega)\sin\omega}{\omega^4}$$

Hence

$$\hat{g}(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{2\omega^2\cos\omega + (\omega^3 - 2\omega)\sin\omega}{\omega^4}$$

- (3) [6 Pts] Let

$$f(t) = \begin{cases} 1 & 0 \le t \le 2 \\ 0 & \text{otherwise;} \end{cases} \qquad g(t) = \begin{cases} t & 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $h(t) = (f * g)(t)$.
(b) Sketch the graph of f, g, h over the interval $[-1, 4]$.

SOLUTION

1. If $t < 0$ or $t > 3$, $h(t) = 0$
2. If $0 \le t < 1$, $h(t) = \int_0^t (t - x)\,dx = (tx - \frac{x^2}{2})\Big|_0^t = \frac{t^2}{2}$
3. If $1 \le t < 2$, $h(t) = \int_{t-1}^t (t - x)\,dx = (tx - \frac{x^2}{2})\Big|_{t-1}^t = \frac{1}{2}$ (area of triangle)
4. If $2 \le t \le 3$, $h(t) = \int_{t-1}^2 (t - x)\,dx = (tx - \frac{x^2}{2})\Big|_{t-1}^2 = -\frac{1}{2}(t-1)(t-3)$