Math 6320 – Fall 2012 Name:

Final Exam

Note that you are supposed to work on your own for this assignment. Please, write clearly and justify all your steps, to get proper credit for your work.

(1)[7 Pts] Let f be a Lebesgue measurable function on $\mathbb{R}^n$. Define

$$\omega(t) = \lambda\{x \in \mathbb{R}^n : |f(x)| > t\}, \quad t \geq 0.$$

Prove that

(a) $\lim_{s\to t^+} \omega(s) = \omega(t)$, for each $t \geq 0$.

(b) If $f \in L^1(\mathbb{R}^n)$, then $\lim_{s\to t^-} \omega(s) = \lambda\{x \in \mathbb{R}^n : |f(x)| \geq t\}$, for each $t \geq 0$.

(c) Note that $\omega(t)$ is not continuous. In fact, you can find an example of an $f \in L^1(\mathbb{R})$ such that $\lim_{s\to t^+} \omega(s) = \omega(t)$ but $\lim_{s\to t^-} \omega(s) \neq \omega(t)$

(d) We have that
$$\int_0^\infty \omega(t)\, dt = \int_{\mathbb{R}^n} |f(x)|\, dx,$$
so that $f \in L^1(\mathbb{R}^n)$ if and only if $\omega \in L^1(\mathbb{R})$.

(2)[3 Pts] Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set and let (f_k) be a sequence of non-negative Lebesgue measurable functions on E satisfying $\lim f_k = f$ a.e.

(a) Show that if $\int_E f d\lambda < \infty$ and

$$\lim_k \int_E f_k\, d\lambda = \int_E f\, d\lambda,$$

then

$$\lim_k \int_A f_k\, d\lambda = \int_A f\, d\lambda,$$

for every Lebesgue measurable set $A \subset E$.

(b) Show that the statement above can fail if $\int_E f d\lambda = \infty$.

(3)[3 Pts] Let $f \in L^1(\mathbb{R})$. Show that the integral

$$F(x) = \int_0^x f(t)\, dt, \quad x \in \mathbb{R},$$

is uniformly continuous on $\mathbb{R}$.

(4)[6 Pts] Prove each the following statements or disprove it by producing a counterexample.

(a) If $f \in C_0(\mathbb{R})$ (i.e., f is continuous and vanish at $\pm\infty$) then $f \in L^1(\mathbb{R})$.

(b) If $f \in L^1(\mathbb{R})$ is uniformly continuous, then $\lim_{x\to\infty} f(x)$ exists and is equal to 0.

(c) If $f \in L^1(\mathbb{R})$ and $\lim_{x\to\infty} f(x) = a$, then $a = 0$.

(d) If $f \in L^1([0,1])$ then

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx = 0.$$